Math 4318 **Practice Problems for Midterm Exam 2** **Fall 2019**

1) Let $g : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Define

$$G : C^0([a,b]) \to C^0([a,b]) : f \mapsto g \circ f.$$

If $C^0([a,b])$ has the sup-norm show that G is uniformly continuous.

2) Show that if $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions that converge uniformly to f and the sequence f_n' converges uniformly to g then $f' = g$.

3) Let $\mathcal{F}$ be a family of equicontinuous functions from an interval $[a,b]$ to $\mathbb{R}$ that are bounded in the sup-norm. Define

$$F(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

for all $x \in [a,b]$. Show that F is continuous.

4) a) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $b \in \mathbb{R}^m$. Define $f(x) = Lx + b$. What is the derivative of f at $y \in \mathbb{R}^n$?

b) Using the definition prove that your answer from a) is the derivative of f.

c) Compute the derivative of $f(x,y,z) = (xy^2, x+y, xy, 5+x)$.

5) For a fixed $c \in [a,b]$ define the function

$$\delta_c : C^0([a,b]) \to \mathbb{R} : f \mapsto f(c).$$

If we give $C^0([a,b])$ the usual sup-norm $\|\cdot\|_\infty$, then show that δ_c is continuous.

6) Show that for any function $f \in C^1([a,b])$ there is a sequence of polynomials p_n that converge to f in the $\|\cdot\|_{C^1}$ norm (that is the p_n converge uniformly to f and p_n' converge uniformly to f'). Hint: use the Weierstrass Theorem to approximate f' first.

7) a) Give and example of a sequence of functions $\{f_n\}$ on the interval $[a,b]$ that are integrable and converge point-wise to a function f that is also integrable but where

$$\int_a^b f(x)\,dx \neq lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

b) Show that if $f_n \to f$ uniformly on $[a,b]$ then f must be integrable on $[a,b]$. (You only need to show f is integrable, but not anything about the integral.)

8) a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $f(x) = g(x) + c$ where $c \in \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\|g(x)\| \leq M\|x\|^2$$

for some constant M. Use the definition of derivative to prove that $Df(0) = 0$.

b) Compute the derivative of $f(x,y,z) = (x^2 z, 3x + 4z)$.

9) Are the following statements true or false (you don't have to justify your answer).

1. Smooth functions are analytic.
2. If a sequence of function $\{f_n\}$ that are integrable on $[a, b]$ and converge to f points-wise then f is integrable.
3. If the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ exists then all the partial derivatives of the coordinate functions exist and are continuous.
4. If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c$ then it defines an differentiable function on the interval $(-|c|, |c|)$.
5. Given any continuous function $f : [5, 10] \to \mathbb{R}$ there is a sequence of polynomials that converge to f uniformly on $[5, 10]$.
6. If all the directional derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}$ exist and are continuous then the derivative of f exists.
7. A continuous function $f : \mathbb{R} \to \mathbb{R}$ must be differentiable somewhere.
8. A Lipschitz map must be differentiable everywhere.
9. Cauchy sequences must converge in a Banach space.