Math 4318 - Spring 2011 Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 3, 5, 6, 11, 12, 16, 19, 20. Due: In class on January 25

- 1. Compute the derivative of $f(x) = x^n$ in two ways. First use the binomial theorem and the definition of derivative. Then use the product rule and induction.
- 2. Compute the first 3 derivatives of $f(x) = |x|^3$.
- 3. Let

$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational.} \end{cases}$$

Show that f is differentiable at 0 and compute f'(0).

4. For real numbers a and b > 0 define the function $f: [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^a \sin |x|^{-b} & x \neq 0\\ 0 & x = 0. \end{cases}$$

Prove that

- (a) f is continuous if and only if a > 0.
- (b) f'(0) exist if and only if a > 1.
- (c) f' is bounded if and only if $a \ge 1 + b$.
- (d) f' is continuous if and only if a > 1 + b.
- 5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and that f(a) = 0. If g(x) = |f(x)| show that g is differentiable at a if and only if f'(a) = 0.
- 6. Recall that a function f is even if f(-x) = f(x) for all x and odd if f(-x) = -f(x) for all x. If f is an even function then show that f' is an odd function.
- 7. Show that $\sqrt{n+1} \sqrt{n} \to 0$ as $n \to \infty$.
- 8. Use the Mean Value Theorem to show that for all $x \in \mathbb{R}$ we have

$$e^x \ge 1 + x$$

with equality if and only if x = 0. Interpret this in terms of the graphs of the functions.

- 9. Let $h : \mathbb{R} \to \mathbb{R}$ be a function with bounded derivative (that is there is some constant M such that $|h'(x)| \leq M$ for all x). For a fixed $\epsilon > 0$ consider the function $f(x) = x + \epsilon h(x)$. Prove that f is injective if ϵ is small enough.
- 10. Suppose that f is defined and differentiable on some interval containing c. Show that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$

Give and example that shows that the limit on the right hand side might exist even if the derivative of f at c does not exits.

11. Suppose that f is defined and twice differentiable in some interval containing c. Show that f(c, t, k) = f(c, t) = 2f(c)

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}$$

Given and example that shows the limit on the right hand side might exist even if the second derivative of f at c does not exits. **Hint:** L'Hopital and the previous problem (which you should do but don't have to write up).

12. Suppose that $f:(a,b) \to \mathbb{R}$ and that f'' exists everywhere. Then show that $f''(x) \ge 0$ for all $x \in (a,b)$ if and only if f is **convex** by which we mean

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y)$$

for all $x, y \in (a, b)$ and $t \in [0, 1]$. **Hint**: Problem 11.

13. We call a function $f : (a, b) \to \mathbb{R}$ Hölder continuous of order α (or α -Hölder continuous) for $\alpha > 0$ if there is some constant M such that for all $x, y \in (a, b)$ we have

$$|f(x) - f(y)| \le M|x - y|^{\alpha}$$

Notice that function that is Hölder continuous of order 1 is Lipschitz.

- (a) Show that a function that is α -Hölder continuous on (a, b) is uniformly continuous on (a, b).
- (b) Show that a function that is α -Hölder continuous on (a.b) for $\alpha > 1$ is constant. **Hint:** First show that such a function is differentiable.
- 14. Let f be a twice differentiable function on $(0, \infty)$. If f'' is bounded and $f(x) \to 0$ as $x \to \infty$ then show that $f'(x) \to 0$ as $x \to \infty$.
- 15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $f'(x) \neq 1$ for all x then show that f has at most one fixed point. (A **fixed point** of a function f is a point x such that f(x) = x.)
- 16. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. If there is a constant $C \in (0, 1)$ such that |f'(x)| < C for all x then show that f does have a fixed point. Show that it is not sufficient to assume that |f'(x)| < 1 to guarantee a fixed point by considering the function

$$f(x) = x + \frac{1}{1 + e^x}.$$

Hint: When C exists let x_1 be any point and set $x_n = f(x_{n-1})$. What can you say about $\lim x_n$? Does it exits?

- 17. Suppose that $f : [-1,1] \to \mathbb{R}$ is three times differentiable. If f(-1) = 0, f(0) = 0, f(1) = 1 and f'(0) = 0, then show that there is some point $x \in (-1,1)$ at which $f^{(3)}(x) \ge 3$. **Hint:** Look at the second order Taylor polynomial about 0 and consider the Remainder terms when you evaluate at ± 1 .
- 18. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If f(a) = 0 and $|f'(x)| \le C|f(x)|$ for some constant C then show that f(x) = 0 for all $x \in [a, b]$.
- 19. Let f be a twice differentiable function on the interval (a, ∞) . Show that

$$\left(\sup_{x\in(a,\infty)}\{|f'(x)|\}\right)^2 \le 4\left(\sup_{x\in(a,\infty)}\{|f(x)|\}\right)\left(\sup_{x\in(a,\infty)}\{|f''(x)|\}\right).$$

Notice that this says we can bound the first derivative of f in terms of f and the second derivative. **Hint:** Consider Taylor polynomial expanded about x evaluated at x + h to get a quadratic equation in h.

- 20. Suppose f is n times continuously differentiable on some interval (a, b) that contains c. If $f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$, then show that
 - (a) If n is even and $f^{(n)}(c) > 0$, then f has a relative minimum at c.
 - (b) If n is even and $f^{(n)}(c) < 0$, then f has a relative maximum at c.
 - (c) If n is odd, then f has neither a relative minimum or a relative maximum at c.

This, of course, is a large generalizations of the "second derivative test" you learned in calculus for determining if a critical point is a max or a min.