Math 4318 - Spring 2011
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 5, 6, 7, 8, 14.* **Due: In class on February 10**

1. Compute $\int_1^3 x^2\,dx$ using the definition of integral. **Hint:** you can use either the Riemann or Darboux integral (since we know they are the same for bounded continuous functions on closed intervals). But one of them will probably be easier to compute.
   Solution: We will used the Darboux integral. Notice that the function x^2 is increasing on $[1,3]$ so the maximum of x^2 on any subinterval of $[1,3]$ will be its value at the right hand end point and similarly its minimum will be its value at the left hand end point. Let $\mathcal{P}$ be the partition of $[1,3]$ with n equal intervals. That is $x_0 = 1, x_1 = 1+\frac{2}{n}, x_2 = 1+\frac{4}{n}, \ldots, x_n = 3$. Now we have

$$\begin{aligned} U(x^2,\mathcal{P}) &= \sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\frac{2}{n} \\ &= \sum_{i=1}^{n} n^{-3}(2n^2+8ni+8i^2) \\ &= n^{-3}\left(2n^2 n + 8n\frac{n(n+1)}{2} + 8\frac{n(n+1)(2n+1)}{6}\right). \end{aligned}$$

So for any n the upper Darboux integral is less than or equal to the last expression in the equation above. Thus it is also less than or equal to the limit of this expression as $n\to\infty$. That is

$$\overline{\int_1^3} x^2\,dx \le \frac{26}{3}.$$

Similarly

$$\begin{aligned} L(x^2,\mathcal{P}) &= \sum_{i=1}^{n}\left(1+\frac{2(i-1)}{n}\right)^2\frac{2}{n} \\ &= \sum_{i=1}^{n} n^{-3}(2n^2+8ni-8n+8i^2-16i+8) \\ &= n^{-3}\left((2n^2-8n+8)n + (8n-16)\frac{n(n+1)}{2} + 8\frac{n(n+1)(2n+1)}{6}\right). \end{aligned}$$

So for any n the lower Darboux integral is greater than or equal to the last expression in the equation above. Thus it is also greater than or equal to its limit as $n\to\infty$. That is

$$\underline{\int_1^3} x^2\,dx \ge \frac{26}{3}.$$

Thus

$$\frac{26}{3} \le \underline{\int_1^3} x^2\,dx \le \overline{\int_1^3} x^2\,dx \le \frac{26}{3}.$$

So the upper and lower integrals are the same. Thus x^2 is integrable with integral

$$\int_1^3 x^2\,dx = \frac{26}{3}.$$

2. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose that for every Riemann integrable function $g : [a, b] \to \mathbb{R}$ we have $\int_a^b f(x)g(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.
Solution: Suppose $f(x) \neq 0$ for some $x \in [a, b]$, say $x = c$. Since f is continuous there is an open interval (u, v) containing a such that $f(x) \neq 0$ for all $x \in (u, v)$. (Hopefully this is clear but if not notice that we can find an $\delta > 0$ such that $|f(x) - f(a)| < |f(a)|/2$ for all $|x - c| < \delta$ with $x \in [a, b]$. Now take $u = a - \delta$ and $v = a + \delta$.) Now take $g(x) = \frac{1}{f(x)}$ for $x \in (u, v)$ and 0 for other x. Since g is piecewise continuous it is integrable on $[a, b]$. Notice that

$$\int_a^b g(x)f(x)\,dx = \int_u^v 1\,dx = v - u \neq 0.$$

So if $f(x)$ is not equal to 0 for all $x \in [a, b]$ there is some integrable g such that $\int_a^b f(x)g(x)\,dx \neq 0$, which is the contrapositive of what was to be proved.
3. Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

and $h : [0, 1] \to \mathbb{R}$ be 1 for x rational and 0 for x irrational. Find a Riemann integrable function $g : [0, 1] \to \mathbb{R}$ so that $g \circ f = h$. Notice that this shows that the composition of two integrable functions need not be integrable.
Solution: Let $g(x) = 1$ for $x \neq 0$ and 0 for $x = 0$. Notice that g has one discontinuity at $x = 0$ and hence g is integrable. Now $g \circ f(x) = h(x)$.
4. Prove the mean value theorem for integrals: If f is continuous on $[a, b]$ there there is a $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Solution: Let $F(x) = \int_a^x f(t)\,dt$. Recall $F'(x) = f(x)$ since f is continuous at all $x \in [a, b]$. The mean value theorem for derivatives says that

$$\int_a^b f(x)\,dx = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a)$$

for some $c \in (a, b)$.
5. Let f and g be continuous on $[a, b]$. If $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ then show there is some $c \in [a, b]$ such that $f(c) = g(c)$.
Solution: From the last problem there is some $c \in (a, b)$ such that

$$0 = \int_a^b f(x) - g(x)\,dx = (f(c) - g(c))(b - a).$$

Thus $f(c) = g(c)$.
6. Compute the first derivatives of the following functions (and carefully justify your computations).
   (a) $F(x) = \int_0^{\sin x} \cos t^2\,dt$
   (b) $G(x) = \int_x^{x^2} \sqrt{1 - t^2}\,dt$
   (c) $H(x) = \int_0^x xe^{t^2}\,dt$ (**Hint:** be careful on this one!)

Solution: Let $\phi(x) = \int_0^x \cos t^2 \, dt$ and $f(x) = \sin x$. Clearly $F(x) = \phi \circ f(x)$. So by the chain rule $F'(x) = \phi'(f(x))f'(x)$ Since $\cos t^2$ is continuous we have $F'(x) = \cos t^2$ and we also know $f'(x) = \cos x$. THus

$$F'(x) = \big(\cos(\sin^2 x)\big) \cos x.$$

Similarly notice that $G(x) = \int_x^0 \sqrt{1-t^2} \, dt + \int_0^{x^2} \sqrt{1-t^2} \, dt$. Thus if $\psi(x) = \int_0^x \sqrt{1-t^2} \, dt$ and $g(x) = x^2$ we see that $G(x) = \phi \circ g(x) - \phi(x)$. We know $\sqrt{1-t^2}$ is continuous so the fundamental theorem of calculus says $\phi'(x) = \sqrt{1-x^2}$. (Notice that for the integral to make sense we need $x, x^2 \in [-1, 1]$, but in that range we have continuity of the integrand.) We also know that $g'(x) = 2x$. Thus the chain rule gives

$$G'(x) = 2x\sqrt{1-x^4} - \sqrt{1-x^2}.$$

Finally let $\eta(x) = \int_0^x e^{t^2} \, dt$. Notice that $H(x) = x\eta(x)$. Since e^{t^2} is continuous we know that $\eta'(x) = e^{x^2}$. Now the product rule gives

$$H'(x) = xe^{x^2} + \int_0^x e^{t^2} \, dt.$$

7. If f is continuous on $[a, b]$ and $\int_a^x f(t) \, dt = \int_x^b f(t) \, dt$ for all $x \in [a, b]$ then show that $f(x) = 0$ for all $x \in [a, b]$.
Solution: Suppose that $f(c) \neq 0$ for some $c \in [a, b]$. With out loss of generality we assume that $f(c) > 0$. Then as we argued in Problem 2, continuity implies there is some interval (d, e) that contains c for which $f(x) > \epsilon$ in (d, e) for some ϵ. We know that $\int_d^e f(t) \, dt \geq \epsilon(e - d)$. Using the hypothesis twice we see

$$\begin{aligned} \int_a^d f(t) \, dt &= \int_d^b f(t) \, dt = \int_d^e f(t) \, dt + \int_e^b f(t) \, dt \\ &= \int_d^e f(t) \, dt + \int_a^e f(t) \, dt = \int_d^e f(t) \, dt + \int_a^d f(t) \, dt + \int_d^e f(t) \, dt. \end{aligned}$$

Thus we see that $\int_d^e f(t) \, dt = 0$. This contradicts that the integral is greater than $\epsilon(e - d)$. Thus $f(x)$ cannot be non-zero for any $x \in [a, b]$.

8. Prove the integral version of the Taylor remainder: Suppose that f and its first $n + 1$ derivatives are continuous on $[a, b]$ and $c \in (a, b)$. For each $x \in [a, b]$ we have that

$$f(x) = f(c) + f'(c)(x - c) + f''(c)/2(x - c)^2 + \ldots + \frac{f^n(c)}{n!}(x - c)^n + R_n$$

where

$$R_n = \frac{1}{n!} \int_c^x (x - t)^n f^{(n+1)}(t) \, dt.$$

Hint: Use integration by parts and induction.
Solution: Assuming $x > c$ (the other case being similar). Notice that for $n = 1$, integrating by parts we have

$$\begin{aligned} R_1 &= \int_c^x (x - t) f^{(2)}(t) \, dt = (x - t) f'(t)|_c^x + \int_c^x f'(t) \, dt \\ &= -(x - c) f'(c) - f(t)|_c^x = -f'(c)(x - c) + f(x) - f(c). \end{aligned}$$

Rearraging the terms gives the formula for $n = 1$. Now inductively assume the formula is true for $n - 1$. We now establish the formula for n. Integrating by parts we see that

$$\begin{aligned} R_n &= \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\, dt \\ &= \frac{1}{n!} \left((x-t)^n f^{(n)}(t)|_c^x + \int_c^x n(x-t)^{n-1} f^{(n)}(t)\, dt \right) \\ &= -\frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{1}{(n-1)!} \int_c^x (x-t)^{n-1} f^{(n)}(t)\, dt \\ &= -\frac{f^{(n)}(c)}{n!}(x-c)^n + R_{n-1}. \end{aligned}$$

By induction we know

$$R_{n-1} = f(x) - \left(f(c) + f'(c)(x-c) + f''(c)/2(x-c)^2 + \ldots + \frac{f^{n-1}(c)}{(n-1)!}(x-c)^{n-1} \right).$$

Substituting this into the last equation and rearranging terms yields the desired formula for n.

14. Prove or disprove the following statements:
    (a) $f \in \mathcal{R}([a,b]) \implies |f| \in \mathcal{R}([a,b])$
    (b) $|f| \in \mathcal{R}([a,b]) \implies f \in \mathcal{R}([a,b])$
    (c) $f \in \mathcal{R}([a,b]) \implies f^2 \in \mathcal{R}([a,b])$
    (d) $f^2 \in \mathcal{R}([a,b]) \implies f \in \mathcal{R}([a,b])$
    (e) $f^3 \in \mathcal{R}([a,b]) \implies f \in \mathcal{R}([a,b])$
    (f) $f^2 \in \mathcal{R}([a,b])$ and $f(x) \geq 0$ for all $x \in [a,b] \implies f \in \mathcal{R}([a,b])$

Soultion: (a) is TURE. It is a theorem from class. In more detail we know since $g(x) = |x|$ is a continuous function the discontinuities of $|f| = g \circ f$ are a subset of those of f. Thus if f is integrable its discontinuities form a set of measure zero and thus so do the set of discontinuities of $|f|$. We are now done by the Riemann-Lebesgue theorem.
(b) is FALSE. Indeed let $f(x)$ be a function on $[0,1]$ that is 1 for rational numbers and -1 for irrational numbers. Then $|f|$ is a constant function and hence integrable. But since f is discontinuous everywhere the Reimann-Lebesgue theorem says it is not integrable.
(c) is TRUE. The product of integrable functions is integrable by a theorem from class.
(d) is FALSE. Use the same counter example as in part (b).
(e) is TRUE. Since $g(x) = x^{1/3}$ is continuous on $\mathbb{R}$ we know that $g \circ h$ is integrable if h is. Thus since f^3 is integrable we know that $g \circ f^3 = f$ is integrable.
(f) is TRUE. We know that $g(x) = x^{1/2}$ is continuous on $[0, \infty)$. So (since f^2 has image in the domain of g) as above $g \circ f^2$ will be integrable since f^2 is. But notice that $g \circ f^2 = |f|$ and since $f(x) \geq 0$ for all $x \in [a,b]$ we see that $|f(x)| = f(x)$. Thus $f = g \circ f^2$ is integrable.