Math 4318 - Spring 2011 Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 1, 2, 4, 5, 7, 8, 9, 10. Due: In class on February 24

1. Let

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0\\ 0 & x = 0 \end{cases}$$

Show that f'(0) = 1 and that f is not strictly increasing in any neighborhood of 0. (Notice that this shows that a function have have positive derivative at a point without being increasing there. What does this say about the continuity of f' near 0?).

2. Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function. Define the function $F : [c, d] \to \mathbb{R}$ by

$$F(t) = \int_{a}^{b} f(x,t) \, dx.$$

Show that F is a continuous function.

3. With the notation as in the previous problem suppose that $f_t = \frac{\partial f}{\partial t}$ is defined and continuous for all points in D (that is for each point (x_0, t_0) think of $f(x_0, t)$ as a function of just t and take its derivative with respect to t). Then the function F from the last problem is differentiable on [c, d] and

$$F'(t) = \int_a^b f_t(x,t) \, dx$$

4. Use the ideas in the last problem to integrate

$$\int_0^1 \frac{x^t - 1}{\ln x} \, dx.$$

You can use that $\frac{d}{dt}a^t = (\ln a)a^t$. Hint: First notice that this is not an improper integral (the integrand is continuous on [0, 1]. Think of the integral as a function f(t). Compute the derivative and then try to recover f(t).

- 5. Let $f_n = \frac{x^n}{1+x^n}$ on the interval [0,2]. Determine what function the f_n converge to. Is the convergence uniform?
- 6. Find a sequence of functions that are everywhere discontinuous on [0, 1] but converge uniformly to a continuous function.
- 7. Suppose that $\{f_n\}$ is a sequence of bounded functions on a set $A \subset \mathbb{R}$ that converge uniformly on A to f. Show that if each of the f_n are bounded then f is bounded.
- 8. Consider $f_n = \frac{nx}{1+nx^2}$ for $x \in [0, \infty)$. Show that the f_n are bound. Let f be the point wise limit of the $\{f_n\}$. Show that f is not bounded. (Form this and the last problem we see that the f_n do not converge uniformly to f.) Let $\{f_n\}$ be a sequence of functions on a set $S \subset \mathbb{R}$. We say the sequence is **equicon-tinuous** on S if for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all
- |x y| < δ with x and y in S and for all n.
 9. Show that if {f_n} is an equicontinuous sequence of functions converges point wise to f, then f is uniformly continuous.
- 10. Show that if $\{f_n\}$ is a sequence of continuous functions that converge uniformly on a compact set, then the sequence is equicontinuous on that set.