Math 4318 - Spring 2011
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 4, 5, 7, 8, 9, 10.* **Due: In class on February 24**

1. Let
$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$
Show that $f'(0) = 1$ and that f is not strictly increasing in any neighborhood of 0. (Notice that this shows that a function have have positive derivative at a point without being increasing there. What does this say about the continuity of f' near 0?).
Solution: To compute the derivative at 0 we look at the difference quotient:
$$\frac{f(0+h) - f(0)}{h} = \frac{h + 2h^2 \sin(1/h)}{h} = 1 + 2h \sin(1/h).$$
We claim that the limit of this is 1 as h goes to zero. To see this consider
$$|(1 + 2h\sin(1/h)) - 1| = |2h \sin(1/h)| \leq 2|h|.$$
Thus given any $\epsilon > 0$ let $\delta = \epsilon/2$ and we see that if $|h - 0| < \delta$ then
$$\left| \frac{f(0+h) - f(0)}{h} - 1 \right| \leq 2|h| < 2\delta = \epsilon.$$
That is $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 1$.
Consider the points $x_n = \frac{1}{\pi/2+2n\pi}$ and $y_n = \frac{1}{-\pi/2+2n\pi}$. Notice that $x_n < y_n$. We claim that $f(y_n) < f(x_n)$. This will show that f is not increasing between y_n and x_n, but since $y_n \to 0$ and $x_n \to 0$ as $n \to \infty$ we see that in any arbitrary neighborhood of 0 the function is not increasing (since it will contain infinitely many of the x_n and y_n). We now compute
$$\begin{aligned} f(x_n) - f(y_n) &= \left(\frac{1}{(\pi/2 + 2n\pi)} + 2\frac{1}{(\pi/2 + 2n\pi)^2} \right) - \left(\frac{1}{(-\pi/2 + 2n\pi)} - 2\frac{1}{(-\pi/2 + 2n\pi)^2} \right) \\ &= \frac{1}{(\pi/2 + 2n\pi)} - \frac{1}{(-\pi/2 + 2n\pi)} + 2\left(\frac{1}{(\pi/2 + 2n\pi)^2} + \frac{1}{(-\pi/2 + 2n\pi)^2} \right) \\ &= -\frac{\pi}{((2n\pi)^2 - (\pi/2)^2)} + 4\left(\frac{(\pi/2)^2 + (2n\pi)^2}{(\pi/2 + 2n\pi)^2(-\pi/2 + 2n\pi)^2} \right) \\ &= \frac{4((\pi/2)^2 + (2n\pi)^2) - \pi((2n\pi)^2 - (\pi/2)^2)}{((2n\pi)^2 - (\pi/2)^2)^2} \\ &= \frac{(4-\pi)(2n\pi)^2 + (4+\pi)(\pi/2)^2}{(\pi/2)^2 + (2n\pi)^2} > 0. \end{aligned}$$

2. Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function. Define the function $F : [c, d] \to \mathbb{R}$ by
$$F(t) = \int_a^b f(x, t)\, dx.$$
Show that F is a continuous function.

Solution: Since f is continuous on a compact set it is uniformly continuous. Now given any $\epsilon > 0$ there is a $\delta > 0$ such that $|(x,t)-(y,s)| < \delta$ implies that $|f(x,t)-f(y,s)| < \epsilon/(b-a)$. Thus for $|t-s| < \delta$ we have

$$|F(t)-F(s)| = \left|\int_a^b f(x,t)-f(x,s)\,dx\right| \le \int_a^b |f(x,t)-f(x,s)|\,dx$$
$$\le \int_a^b \epsilon/(b-a)\,dx = (b-a)(\epsilon/(b-a)) = \epsilon.$$

Thus F is continuous (in fact uniformly continuous).

4. Use the ideas in the last problem to integrate

$$\int_0^1 \frac{x^t-1}{\ln x}\,dx.$$

You can use that $\frac{d}{dt}a^t = (\ln a)a^t$. Hint: First notice that this is not an improper integral (the integrand is continuous on $[0,1]$. Think of the integral as a function $f(t)$. Compute the derivative and then try to recover $f(t)$.
Solution: Let

$$F(t) = \int_0^1 \frac{x^t-1}{\ln x}\,dx.$$

Then from the previous homework problem we know that

$$F'(t) = \int_0^1 \frac{x^t \ln x}{\ln x}\,dx = \int_0^1 x^t\,dx = \frac{1}{t+1}x^{t+1}|_0^1 = \frac{1}{t+1}.$$

Thus the fundamental theorem of calculus gives us that

$$F(t)-F(0) = \int_0^t F'(s)\,ds = \int_0^t \frac{1}{s+1}\,ds = \ln(s+1)|_0^t = \ln(t+1).$$

Notice that $F(0) = \int_0^1 0\,dt = 0$, so we see that $F(t) = \ln(t+1)$. In other words

$$\int_0^1 \frac{x^t-1}{\ln x}\,dx. = \ln(t+1).$$

5. Let $f_n = \frac{x^n}{1+x^n}$ on the interval $[0,2]$. Determine what function the f_n converge to. Is the convergence uniform?
Solution: If $x \in [0,1)$ then the limit of the numerator is 0 and the limit of the denominator is 1. Thus the limit is 0. If $x = 1$ then each term is 1/2 so the limit is 1/2. If $x \in (1,2]$ then one may easily check that the limit is 1. Thus the limit function is

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1/2 & x = 1 \\ 1 & x \in (1,2]. \end{cases}$$

Since the f_n are continuous and f is not, the convergence cannot be uniform.

7. Suppose that $\{f_n\}$ is a sequence of bounded functions on a set $A \subset \mathbb{R}$ that converge uniformly on A to f. Show that if each of the f_n are bounded then f is bounded.

Solution: By the definition of uniform convergence there is some N such that for all $n \geq N$ we know that

$$|f(x) - f_n(x)| \leq 1$$

for all $x \in A$. Also since f_N is bounded there is some M such that $|f_N(x)| \leq M$ for all $x \in A$. Thus

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M.$$

Thus $M + 1$ is a bound on f.

8. Consider $f_n = \frac{nx}{1+nx^2}$ for $x \in [0, \infty)$. Show that the f_n are bound. Let f be the point wise limit of the $\{f_n\}$. Show that f is not bounded. (Form this and the last problem we see that the f_n do not converge uniformly to f.)

Solution: Notice that for a fixed n we have $f_n(x) \to 0$ as $x \to \infty$. Thus there is some B such that $x \geq B$ implies that $|f_n(x)| < 1$. Now on $[-B, B]$, being a compact set, the continuous function f_n is bounded, say by M. Thus if M' is the maximum of M and 1 we see that f_n is bounded by M'. Thus all the f_n are bounded.

If $x \neq 0$ then $f_n(x)$ clearly limits to $1/x$ as $n \to \infty$. and $f_n(0) = 0$ so for $x = 0$, $f_n(x)$ limits to 0. Given any M there is some $x > 0$ such that $1/x > M$. Thus $f(x) > M$, that is f is not bounded by any M on $[0, \infty)$.

Let $\{f_n\}$ be a sequence of functions on a set $S \subset \mathbb{R}$. We say the sequence is **equicontinuous** on S if for every $\epsilon > 0$ there is some $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $|x - y| < \delta$ with x and y in S and for all n.

9. Show that if $\{f_n\}$ is an equicontinuous sequence of functions converges point wise to f, then f is uniformly continuous.

Solution: Given $\epsilon > 0$ we know there is a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon/3$ for all $|x - y| < \delta$. Also given any x and y there is an N such that $|f_n(x) - f(x)| < \epsilon/3$ and $|f_n(y) - f(y)| < \epsilon/3$ for all $n \geq N$. Thus for any $|x - y| < \delta$ we can find an n we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

That is f is uniformly continuous.

10. Show that if $\{f_n\}$ is a sequence of continuous functions that converge uniformly on a compact set, then the sequence is equicontinuous on that set.

Solution: Let f be the function to which the f_n converge and D be the compact set on which the functions are defined and converge uniformly. Since the convergence is uniform we know that f is continuous. Moreover since we are considering f on a compact set we know f is uniformly continuous. Thus given any ϵ there is some $\delta' > 0$ such that for all $|x - y| \leq \delta'$ we know $|f(x) - f(y)| < \epsilon/3$. There is also an N such that for all $n \geq N$ we know $|f_n(x) - f(x)| \leq \epsilon/3$ for all $n \geq N$ and $x \in D$. Now we see that for any $|x - y| < \delta'$ and $n \geq N$ we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Now for each $n < N$ there is some δ_n such that when $|x - y| < \delta_n$ we have $|f_n(x) - f_n(y)| \leq \epsilon$ (since each f_n is uniformly continuous as it is a continuous function on a compact set). Now let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_{N-1}, \delta'\}$. Then for any n we see that if $|x - y| < \delta$ we have $|f_n(x) - f_n(y)| \leq \epsilon$. That is the f_n are equicontinuous.