Math 4318 - Spring 2011
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 4, 6, 11, 12, 13, 14, 15.* **Due: In class on March 10**

2. Let
$$f_n(x) = \frac{x}{1+nx^2}$$
be a sequence of functions. Show that $\{f_n\}$ converge uniformly to some function f and that
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
for all $x \neq 0$, but that the equality is not true for $x = 0$.
Solution: Let $f(x) = 0$. We claim that $f_n \to f$ uniformly. To see this assume any $\epsilon > 0$ is given. Notice that
$$|f(x) - f_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq |x| < \epsilon$$
if $|x| < \epsilon$. If $|x| \geq \epsilon$ notice that
$$|f(x) - f_n(x)| = \left|\frac{x}{1+nx^2}\right| \leq \left|\frac{x}{x^2 n}\right| = \frac{1}{n|x|} \leq \frac{1}{n\epsilon}$$
We know that $1/(n\epsilon) \to 0$ as $n \to \infty$. Thus there is some N such that $n \geq N$ implies that $1/(n\epsilon) < \epsilon$. Thus we see that if $n \geq N$ we have
$$|f(x) - f_n(x)| < \epsilon$$
for all $x \in \mathbb{R}$. That is $f_n \to f$ uniformly.
Notice that
$$f_n'(x) = \frac{(1+nx^2) - 2x^2 n}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$
For $x \neq 0$ we see the numerator is quadratic in n and the denominator is quartic in n, thus $f_n'(x) \to 0$ as $n \to \infty$. Since $f'(x) = 0$ we see that
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
if $x \neq 0$. If $x = 0$ notice that $f_n'(x) = 1$. So $f_n'(0) \to 1$ as $n \to \infty$ which is not $f'(0) = 0$.

4. Let $f \in C^0([0,1])$. Show that if
$$\int_0^1 x^n f(x)\,dx = 0$$
for all non-negative integers n then $f(x) = 0$.
Hint: Think about the Weierstrass theorem and try to show that $\int_0^1 f^2(x)\,dx = 0$.
Solution: By the Weierstrass theorem there is a sequence of polynomials p_n such that $p_n \to f$ uniformly on $[0,1]$. Since f is continuous on a compact set it is bounded and since the p_n converge uniformly to f we know the p_n are uniformly bounded as well.

Thus the sequence fp_n converges uniformly to f^2 on $[0,1]$ and from a theorem from class we know

$$\int_0^1 f^2(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x)p_n(x)\,dx.$$

But notice that for any polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ we have

$$\int_0^1 p(x)f(x)\,dx = \sum_{k=0}^{n} a_k \int_0^1 x^k f(x)\,dx = 0.$$

Thus $\int_0^1 f(x)p_n(x)\,dx = 0$ for all n and we see that $\int_0^1 f^2(x)\,dx = 0$. Since f^2 is continuous and non-negative we know from an earlier homework that $f^2(x) = 0$ for all $x \in [0,1]$. Thus $f(x) = 0$ for all $x \in [0,1]$.

6. Given two functions $f, g \in \mathcal{R}([a,b])$ define the L^2-inner product to be

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Why is this not an inner product on $\mathcal{R}([a,b])$? Show that this does give an inner product on the set of continuous functions $C^0([a,b])$. (Notice that this also gives a norm on $C^0([a,b])$ by $\|f\|_2 = \sqrt{\langle f, f\rangle}$.)

Solution: Consider the function $f : [a,b] \to \mathbb{R}$ that is zero for all x except $f(a) = 1$. This function is integrable (since it is continuous almost everywhere and bounded) and it is not the zero function but

$$\langle f, f\rangle = 0.$$

Thus the L^2-inner product is not an inner product on $\mathcal{R}([a,b])$.

Notice that for any $f, g, h \in \mathcal{R}([a,b])$ (in particular functions in $C^0([a,b])$) we have

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g, f\rangle$$

and

$$\langle f, g+h\rangle = \int_a^b f(x)(g(x)+h(x))\,dx = \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx = \langle f, g\rangle + \langle f, h\rangle.$$

If $c \in R$ then

$$\langle f, cg\rangle = \int_a^b cf(x)g(x)\,dx = c\int_a^b f(x)g(x)\,dx = c\langle f, g\rangle.$$

In addition we have

$$\langle f, f\rangle = \int_a^b f^2(x)\,dx \geq 0$$

since $f^2(x) \geq 0$ for all x. Lastly if $f \in C^0([a,b])$ and

$$0 = \langle f, f\rangle = \int_a^b f^2(x)\,dx$$

then $f^2(x) = 0$ for all $x \in [a,b]$ by a previous homework problem (since f^2 is continuous). Thus $f(x) = 0$ for all $x \in [0,1]$, that is f is the zero function and we have verified that the L^2 inner product is indeed an inner product on $C^0([a,b])$.

11. If $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on V then a sequence $\{v_n\}$ is Cauchy in the norm $\|\cdot\|_a$ if and only if it is Cauchy in the norm $\|\cdot\|_b$. (Together with the previous problem we see that $(V, \|\cdot\|_a)$ is a Banach space if and only if $(C, \|\cdot\|_b)$ is a Banach space when the norms are equivalent.)
Solution: By the definition of equivalence we have positive constants C and C' such that

$$C\|v\|_a \leq \|v\|_b \leq C'\|V\|_a.$$

Let $\{v_n\}$ be a Cauchy sequence in V with the norm $\|\cdot\|_a$. Then given any $\epsilon > 0$ there is an N such that for all $m, n \geq N$ we have

$$\|v_n - v_m\|_a < \epsilon/C'.$$

Thus we see that

$$\|v_n - v_m|_b \leq C'\|v_n - v_m\|_a < C'(\epsilon/C') = \epsilon,$$

that is $\{v_n\}$ is Cauchy in the norm $\|\cdot\|_b$.
Similarly if we are given a sequence $\{v_n\}$ that is Cauchy with the norm $\|\cdot\|_b$. Then given any $\epsilon > 0$ there is an N such that for all $m, n \geq N$ we have

$$\|v_n - v_m\|_b < \epsilon C.$$

Thus we see that

$$\|v_n - v_m|_a \leq \frac{1}{C}\|v_n - v_m\|_b < \frac{1}{C}(\epsilon C) = \epsilon,$$

that is $\{v_n\}$ is Cauchy in the norm $\|\cdot\|_a$.

12. Let $\|\cdot\|_\infty$ be the sup-norm on $C^1([a,b])$. Is $(C^1([a,b]), \|\cdot\|_\infty)$ a Banach space? In class we saw that $(C^1([a,b]), \|\cdot\|_{C^1})$ is a Banach space (recall $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$). Are the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{C^1}$ equivalent on $C^1([a,b])$?
Solution: Let $f_n : [-1,1] \to \mathbb{R}$ be the function given by $f_n(x) = \sqrt{x^2 + 1/n}$. These functions are all continuously differentiable so $\{f_n\}$ is a sequence in $C^1([-1,1])$. It is clear that this sequence converges point-wise to $f(x) = |x|$. We claim the convergence is also uniform. Indeed given $\epsilon > 0$ there is some N such that $1/\sqrt{N} < \epsilon$. Then for $n \geq N$ we see that

$$\begin{aligned}|f_n(x) - f(x)| &= \sqrt{x^2 + 1/n} - |x| \leq \sqrt{x^2 + 1/N} - |x| \\ &\leq \sqrt{|x|^2 + \frac{2}{\sqrt{N}}|x| + \frac{1}{N}} - |x| = \sqrt{\left(|x| + \frac{1}{\sqrt{N}}\right)^2} - |x| \\ &= |x| + \frac{1}{\sqrt{N}} - |x| = \frac{1}{\sqrt{N}} < \epsilon.\end{aligned}$$

(Here we used that $f_n(x) > f(x)$ for all x and the fact that the square root function is increasing (which is hopefully obvious)).
Since $f_n \to f$ uniformly we see that $\{f_n\}$ converges to f in the norm $\|\cdot\|_\infty$. Thus $\{f_n\}$ is Cauchy in the norm $\|\cdot\|_\infty$. But since $f \notin C^1([-1,1])$ we see that a Cauchy sequence in $C^1([-1,1])$ in the sup norm does not necessarily converge to a function in $C^1([-1,1)$. Thus $(C^1([-1,1), \|\cdot\|_\infty)$ is not a Banach space.
The norms $\|\cdot\|_\infty$ and $\|\cdot\|_{C^1}$ cannot be equivalent by the previous two exercises.

13. Consider $\mathcal{R}([a,b])$ with the sup norm $\|\cdot\|_\infty$ and $C^0([a,b])$ with the sup norm. Define a function

$$I : \mathcal{R}([a,b]) \to C^0([a,b])$$

by $I(f)(x) = \int_a^x f(s)\,ds$. Show that I is a uniformly continuous function (that is given any $\epsilon > 0$ there is a $\delta > 0$ such that for all $\|f-g\|_\infty < \delta$ we have $\|I(f) - I(g)\|_\infty < \epsilon$).
Solution: Given $\epsilon > 0$ let $\delta = \epsilon/(2(b-a))$. Then if $\|f-g\|_\infty < \delta$ we see that

$$\begin{aligned}|I(f)(x) - I(g)(x)| &= \left|\int_a^x f(s) - g(s)\,ds\right| \le \int_a^x |f(s) - g(s)|\,ds \\ &\le \int_a^b |f(s)-g(s)|\,ds < \int_a^b \delta\,dx = \delta(b-a) = \epsilon/2.\end{aligned}$$

Thus

$$\|I(f) - I(g)\|_\infty = \sup_{x\in[a,b]} |I(f)(x) - I(g)(x)| \le \epsilon/2 < \epsilon.$$

14. With the notation from the last problem assume that $\{f_n\}$ is a bonded sequence in $\mathcal{R}([a,b])$ with the sup norm. Show that $\{I(f_n)\}$ has a convergent subsequence (that is converges uniformly to some function f on $[a,b]$).
Solution: Since the sequence $\{f_n\}$ is bounded we know there is some M such that $|f_n(x)| \le \|f_n\|_\infty < M$ for all M. That is the sequence is uniformly bounded. Notice that

$$|I(f_n)(x)| = \left|\int_a^x f_n(s)\,ds\right| \le \int_a^x |f_n(s)|\,ds \le \int_a^x M\,ds \le \int_a^b M\,dx = M(b-a).$$

That is, the sequence of continuous function $\{I(f_n)\}$ is uniformly bounded by $M(b-a)$. We claim that the sequence is also equicontinuous. Indeed for any $\epsilon > 0$ we notice that if $\delta = \epsilon/M$ then when $|x-y| < \delta$ we have

$$\begin{aligned}|I(f_n)(x) - I(f_n)(y)| &= \left|\int_a^x f(s)\,ds - \int_a^y f(s)\,ds\right| = \left|\int_x^y f(s)\,ds\right| \\ &\le \left|\int_x^y |f(s)|\,ds\right| \le M|x-y| < M\delta = \epsilon.\end{aligned}$$

Thus we see that $\{I(f_n)\}$ is a sequence of continuous functions on $[a,b]$ that are uniformly bounded and equicontinuous. So the Arzelà–Ascoli theorem says there is a subsequence that converges uniformly to some function f.

15. Let $\{f_n\}$ be a sequence of functions in $C^1([a,b])$ that are bounded in the norm $\|\cdot\|_{C^1}$. Show that there is a subsequence that converges to a function f in the $\|\cdot\|_\infty$ norm.
Solution: Since $\{f_n\}$ is bounded in the $\|\cdot\|_{C^1}$ norm we know that there is some M such that

$$\|f_n\|_\infty + \|f_n'\|_\infty = \|f_n\|_{C^1} \le M.$$

Thus for all x we have

$$|f_n(x)! \le \|f_n\|_\infty \le \|f_n\|_{C^1} \le M.$$

That is, we know the sequence is uniformly bounded.
In addition we know that for all x we have

$$\|f_n'(x)\| \le \|f_n'\|_\infty \le \|f_n\|_{C^1} \le M.$$

Thus by the mean value theorem we know that for any x, y we have some c between x and y so that

$$|f_n(x) - f_n(y)| = |f'(c)||x - y| \leq M|x - y|$$

Hence if $\epsilon > 0$ then we can take $\delta = \epsilon/M$ and notice that if $|x - y| < \delta$ we have

$$|f_n(x) - f_n(y)| \leq M|x - y| < M\delta = \epsilon.$$

That is we have shown that $\{f_n\}$ is equicontinuous. Since it is also a uniformly bounded sequence of continuous funciotns on a compact interval the Arzelà–Ascoli theorem says there is a subsequence that converges uniformly to some function f. Of course this is the same as converging in the $\| \cdot \|_\infty$ norm.