Math 4318 - Spring 2011 Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 2, 3, 4, 11, 12, 13, 16, 17. Due: In class on March 31

2. For b > 0 and any a define

$$T(f)(x) = a + \int_0^x f(y)e^{-xy} \, dy.$$

Show that $T: C^0([0,b]) \to C^0([0,b])$ is a contraction. Hence show that there is a unique solution to

$$f(x) = a + \int_0^x f(y)e^{-xy} \, dy$$

in $C^0([0,b])$.

Solution: Given two functions $f, g \in C^0([0, b])$ notice that

$$\begin{split} \|T(f) - T(g)\|_{\infty} &= \|\int_{0}^{x} (f(y) - g(y))e^{-xy} \, dy\|_{\infty} \\ &= \sup_{x \in [0,b]} \left\{ \left| \int_{0}^{x} (f(y) - g(y))e^{-xy} \, dy \right| \right\} \\ &\leq \sup_{x \in [0,b]} \left\{ \int_{0}^{x} \|f(y) - g(y)|e^{-xy} \, dy \right\} \\ &\leq \sup_{x \in [0,b]} \left\{ \int_{0}^{x} \|f - g\|_{\infty} e^{-xy} \, dy \right\} \\ &= \|f - g\|_{\infty} \sup_{x \in [0,b]} \left\{ \left| \frac{1}{x} e^{-xy} \right|_{0}^{x} \right| \right\} = \sup \left\{ \frac{1}{x} (1 - e^{-x^{2}}) \right\} \|f - g\|_{\infty}. \end{split}$$

Since $\frac{1}{x}(1-e^{-x^2})$ is a continuous function on a compact interval [0, b] we see that T will be a contraction mapping if $\frac{1}{x}(1-e^{-x^2}) < 1$. That is if $1-x < e^{-x^2}$. To see this notice that by the mean value theorem we have $e^x - e^0 = e^t(x-0)$ for some t between x and 0. Thus we have $e^x > x+1$ for any $x \neq 0$. Now if $x \in (0,1]$ then $1-x \leq 1-x^2 < e^{-x^2}$. If x > 1 then clearly $1-x < 0 < e^{-x^2}$. Now the contraction mapping theorem implies there is a function $f \in C^0([0,b])$ satisfying

$$f = T(f) = a + \int_0^x f(y)e^{-xy} \, dy.$$

3. Let $\phi : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a function that is C^n (that is if you fix either of the variables the function is C^n differentiable with respect to the other variable). Let

$$\Phi(f)(t) = c + \int_a^t \phi(s, f(s)) \, ds$$

(recall this is the "integral operator" used in the proof that ODEs have solutions). Show that if f is a fixed point of Φ then f is has n + 1 continuous derivatives on [a, b]. (You may use the fact that $\frac{d}{dt}\phi(t, f(t)) = \phi_t(t, f(t)) + \phi_x(t, f(t))f'(t)$.)

Notice that this problem says that if f is a solution to the differential equation

$$y' = \phi(t, y) \quad y(t_0) = x_0$$

with $\phi \in C^n$ function then $f \in C^{n+1}$ (where it is defined). Solution: Assume that ϕ is C^n . If $f = \phi(f)$ then

$$f(t) = c + \int_a^t \phi(s, f(s)) \, ds.$$

The fundamental theorem of calculus gives us that

$$f'(t) = \phi(t, f(t))$$

Notice that we are not given that f is continuous, but since the above formula shows the derivative of f exists we know that it is continuous. Since ϕ and f are continuous we see that f'(t) is continuous. Now

$$f''(t) = \phi_t(t, f(t)) + \phi_x(t, f(t))f'(t).$$

Since f', ϕ and f are continuous (and the product and composition of continuous functions is continuous) we see that f'' is continuous. Notice that the kth derivative of f(for $k \leq n+1$) can be expressed as products and compositions of derivatives of (partial derivatives of) ϕ and f of order $\langle k$. (We see this for k = 1, 2 and using the product rule and chain rule easily see that if it is true of k then it is true for k + 1 if $k \leq n$.) Now if $f^{(k)}$ is continuous then $f^{(j)}$ is continuous for all $j \leq k$. Thus $f^{(k+1)}$ is continuous since it can be expressed as a composition and product of continuous functions. So by induction again we see that $f^{(n+1)}$ is continuous.

4. Continuing the previous problem consider the function

$$\phi(t,x) = \begin{cases} t & t \le 1\\ 2-t & t \ge 1. \end{cases}$$

Solve the differential equation $y' = \phi(t, y)$ with y(0) = 1 and show that it is C^1 but not twice differentiable.

Solution: Integrating we see that a solution γ to the differential equation is $\gamma(t) = \frac{1}{2}t^2 + 1$ for $t \leq 1$ and $\gamma(t) = 2t - \frac{1}{2}t^2$ for $t \geq 1$. Notice that γ is continuous and $\gamma' = t$ for $t \leq 1$ and 2 - t for $t \geq 1$. Thus γ' is also continuous. Now $\gamma''(t) = 1$ for t < 1 and $\gamma''(t) = -1$ for t > 1. It is easy to see that $\gamma''(1)$ does not exist since taking a sequence $x_n \to 1$ approaching 1 from the left we see that

$$\frac{\gamma'(x_n) - \gamma'(1)}{x_n - 1} = \frac{-x_n - 1}{x_n - 1} \to -1 \text{ as } n \to \infty.$$

where as a if y_n is a sequence approaching 1 from the left then

$$\frac{\gamma'(y_n) - \gamma'(1)}{y_n - 1} = \frac{y_n - 1}{y_n - 1} \to 1 \text{ as } n \to \infty.$$

Thus γ is C^1 but not C^2 proving that the result of the previous exercise is optimal. 11. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that $||f(x)|| \le M ||x||^2$ for all

The let $f: \mathbb{R}^n$ will and suppose there is a constant M such that $||f(x)|| \leq M ||x||$ for $x \in \mathbb{R}^n$. Prove that f is differentiable at x = 0 and Df(0) = 0. Solution: Notice that f(0) = 0 since ||f(0)|| < 0. Consider

$$\frac{\|f(x) - (f(0) + 0(x - 0))\|}{\|x - 0\|} = \frac{\|f(x)\|}{\|x\|} \le \frac{M\|x\|^2}{\|x\|} = M\|x\|.$$

So as $x \to 0$ we see that $\frac{\|f(x) - (f(0) + 0(x-0)\|}{\|x-0\|} \to 0$. Thus f is differentiable at 0 and its derivative is the linear map 0.

12. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that L is differentiable at all $x \in \mathbb{R}^n$ and compute DL(x).

Solution: The derivative is the best linear approximation to a map and one would expect the best linear approximation to a linear map is itself. So we check that the linear map DL(x) = L satisfies the definition of the derivative of L at x. To this end consider

$$\frac{\|L(y) - (Lx + L(y - x))\|}{\|y - x\|} = \frac{\|L(y) - Lx - (Ly - Lx)\|}{\|y - x\|} = \frac{0}{\|y - x\|} = 0,$$

where the linearity of L was used in the first equality. Thus $\frac{\|L(y) - (Lx + L(y-x))\|}{\|y-x\|} \to 0$ as $y \to x$, from which we see that L is differentiable at x and DL(x) = L.

13. Compute the derivative (that is Jacobian matrix) of

- (a) $f(x,y) = \sin(x^2 + y^3)$
- (b) $g(x, y, z) = (z \sin x, x \sin y)$
- (c) $h(x, y, z) = (x^2, xy)$

Solution: (a) Since all the partial derivatives of f exist and are continuous we know f is differentiable and its derivative (using the standard basis for \mathbb{R}^2 and \mathbb{R}) is

$$Df(x,y) = \begin{bmatrix} 2x\cos(x^2 + y^2) & 2y\cos(x^2 + y^2) \end{bmatrix}.$$

(b) The partial derivatives of the component functions $(z \sin x \text{ and } x \sin y)$ exist and are continuous. Thus g is differentiable with derivative

$$Dg(x, y, z) = \begin{bmatrix} z \cos x & 0 & \sin x \\ \sin y & x \cos y & 0 \end{bmatrix}$$

(c) The component functions all have continuous derivatives so the function h is differentiable and the derivative is given by

$$Dh(x, y, z) = \begin{bmatrix} 2x & 0 & 0 \\ y & x & 0 \end{bmatrix}$$

16. Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ satisfy the conditions that f(0) = (1, 2) and

$$Df(0) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

If in addition $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by g(x, y) = (x + 2y + 1, 3xy) then find $D(g \circ f)(0)$. Solution: From the chain rule we know

$$D(g \circ f)(0) = Dg(f(0))Df(0)$$

To compute Dg we note that its component functions are continuously differentiable and hence we have

$$Dg = \begin{bmatrix} 1 & 3y \\ 2 & 3x \end{bmatrix}.$$

So we have

$$Dg(f(0)) = Dg(1,2) = \begin{bmatrix} 1 & 6\\ 2 & 3 \end{bmatrix}.$$

Finally we have

$$D(g \circ f)(0) = \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 9 \\ 2 & 4 & 9 \end{bmatrix}.$$

17. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. If h(x, y) = f(x, y, g(x, y)) then compute Dh in terms of the partial derivatives of f and g. If h = 0, then write $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in terms of the partial derivatives of f. Solution: Let G(x, y) = (x, y, g(x, y)) so that we see $h(x, y) = (f \circ G)(x, y)$. Assuming

Solution: Let G(x, y) = (x, y, g(x, y)) so that we see $h(x, y) = (f \circ G)(x, y)$. Assuming everything is differentiable then we have

$$Dh = Df(G(x,y))DG(x,y)$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(x,y,g(x,y)) & \frac{\partial f}{\partial y}(x,y,g(x,y)) & \frac{\partial f}{\partial z}(x,y,g(x,y)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(x,y,g(x,y)) + \frac{\partial f}{\partial z}(x,y,g(x,y)) \frac{\partial g}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y,g(x,y)) + \frac{\partial f}{\partial z}(x,y,g(x,y)) \frac{\partial g}{\partial y}(x,y) \end{bmatrix}$$

If h = 0 then Dh = 0 and we see that

$$\frac{\partial f}{\partial x}(x, y, g(x, y)) + \frac{\partial f}{\partial z}(x, y, g(x, y))\frac{\partial g}{\partial x}(x, y) = 0$$

and

$$\frac{\partial f}{\partial y}(x, y, g(x, y)) + \frac{\partial f}{\partial z}(x, y, g(x, y))\frac{\partial g}{\partial y}(x, y).$$

In other words assuming $\frac{\partial f}{\partial z}(x,y,g(x,y))\neq 0$ we have

$$\frac{\partial g}{\partial x}(x,y) = \frac{\frac{\partial f}{\partial x}(x,y,g(x,y))}{\frac{\partial f}{\partial z}(x,y,g(x,y))}$$

and

$$\frac{\partial g}{\partial y}(x,y) = \frac{\frac{\partial f}{\partial y}(x,y,g(x,y))}{\frac{\partial f}{\partial z}(x,y,g(x,y))}.$$