

# Math 4318 - Spring 2011

## Homework 6

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 1, 4, 7, 8, 9, 10, 11, 12. **Due: In class on April 21**

1. A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called homogeneous of degree  $k$  if  $f(tx) = t^k f(x)$ . Show that for a such a function

$$Df(x)(x) = kf(x).$$

Hint: Take the derivative of  $g(t) = f(tx)$ .

2. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfy  $\frac{\partial f}{\partial x_1}(x) = 0$  for all  $x \in \mathbb{R}^n$ . Show that  $f(x)$  only depends on  $x_2, \dots, x_n$ . (Recall we are writing  $x$  in coordinates as  $x = (x_1, \dots, x_n)$ .)
3. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  satisfies  $\|f(t)\| = 1$  for all  $t$ . Then show that  $Df(t) \cdot f(t) = 0$  (where  $\cdot$  is the dot product).
4. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function and let  $B = \{v \in \mathbb{R}^n : \|v\| \leq 1\}$ . If  $f$  is differentiable on the interior of  $B$  and  $f = 0$  on the boundary of  $B$ , then show that there is some point  $x_0$  in the interior of  $B$  such that  $Df(x_0) = 0$ .
5. Compute the second order Taylor polynomial of  $f(x, y) = e^x \cos y$  at  $(0, 0)$ .
6. Let  $B : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$  be a bilinear map. Show that the derivative of  $B$  at  $(x_0, y_0)$  is

$$DB(x_0, y_0)(x, y) = B(x_0, y) + B(x, y_0).$$

7. Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Compute  $D^2L$ .
8. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^p$  be two functions that are twice continuously differentiable. Show that

$$D^2(g \circ f)(x_0)(x, y) = D^2(g(f(x_0)))(Df(x_0)(x), Df(x_0)(y)) + Dg(f(x_0))D^2f(x_0)(x, y).$$

9. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto (x^2 - y^2, 2xy)$ . Show that this function is locally invertible at all points  $(x, y) \neq (0, 0)$ . If we set  $(u, v) = f(x, y)$  (that is  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy$ ), then compute  $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ .
10. In an earlier homework assignment you showed that

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies  $f(0) = 0, f'(0) \neq 0$  but that  $f$  is not locally invertible near 0. How does this not contradict the inverse function theorem?

11. Given the system of equations

$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + 2x + 3z^2, \end{aligned}$$

can we always solve for  $x, y, z$  in terms of  $u, v, w$  near  $(0, 0, 0)$ ? Explain.

12. Show that the equations

$$\begin{aligned} x^2 - y^2 - u^3 + v^2 + 4 &= 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0 \end{aligned}$$

determine functions  $u(x, y)$  and  $v(x, y)$  near  $x = 2, y = -1$  such that  $u(2, -1) = 2$  and  $v(2, -1) = 1$ . Compute  $\frac{\partial u}{\partial x}$ .