Math 4318 - Spring 2011
Homework 6

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 4, 7, 8, 9, 10, 11, 12.* **Due: In class on April 21**

1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called homogeneous of degree k if $f(tx) = t^k f(x)$. Show that for a such a function
$$Df(x)(x) = kf(x).$$
Hint: Take the derivative of $g(t) = f(tx)$.
Solution: Let $g(t) = f(tx) = t^k f(x)$. So $g'(t) = Dg(t) = D(t^k f)(x) = kt^{k-1}f(x)$ and $Dg(1) = kf(x)$. In addition, using the chain rule we have $Dg(t) = Df(tx)(x)$, thus $Df(x)(x) = kf(x)$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $B = \{v \in \mathbb{R}^n : \|v\| \leq 1\}$. If f is differentiable on the interior of B and $f = 0$ on the boundary of B, then show that there is some point x_0 in the interior of B such that $Df(x_0) = 0$.
Solution: If $f(x)$ is constant then $Df(x) = 0$ for all x and we are done. If $f(x)$ is not constant then there is some point x_1 for which $f(x_1)$ is larger than or less than 0. Assume without loss of generality that $f(x_1) > 0$. Thus since B is compact (this follows from the Heine-Borel theorem since B is clearly bounded had it is $g^{-1}((-\infty, 1])$ for the continuous function $g(x) = \|x\|$) we know f, being continuous on B, has maximum value at some point x_0 in the interior of B and $f(x_0) > 0$. Since $f(x_0)$ is an extreme value we know by a theorem in class that $Df(x_0) = 0$.

7. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Compute D^2L.
Solution: We know that $DL(x) = L$ for all $x \in \mathbb{R}^n$. That is $DL : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a constant map. Since the derivative of a constant map is the zero linear map we see that $D^2L = 0$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be two functions that are twice continuously differentiable. Show that
$$D^2(g \circ f)(x_0)(x, y) = D^2(g(f(x_0)))(Df(x_0)(x), Df(x_0)(y)) + Dg(f(x_0))D^2f(x_0)(x, y).$$
Solution: We know by the chain rule that $D(g \circ f)(x) = Dg(f(x))Df(x)$. As a map from $\mathbb{R}^n$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ we see that $D(g \circ f)(x)(y) = Dg(f(x))Df(x)(y)$. Notice that we can think of $D(g \circ f)(x)(y)$ as a product of $Dg(f(x))$ and $Df(x)(y)$. Thus using the product rule we have
$$D(D(g \circ f)(x)(y))(z) = D(Dg(f(x))(z)(Df(x)(y)) + Dg(f(x))D(D(f(x)(y))(z).$$
Now notice that $h(x) = Dg(f(x))$ is a linear map $\mathbb{R}^m \to \mathbb{R}^m$. That is $h : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$. So $Dh(x)$ is a linear map $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$ and $Dh(x)(z)$ is a linear map $\mathbb{R}^m \to \mathbb{R}^p$. Using the chain rule we have $Dh(x)(z) = D(Dg(f(x)))(Df(x)(z))$ which is a linear map $\mathbb{R}^m \to \mathbb{R}^p$. Applying this to $Df(x)(y)$ we get $Dh(x)(z)(Df(x)(y)) = D^2(g(f(x)))(Df(x)(z), Df(x)(y))$. Plugging this into the above equation we get
$$D^2(g \circ f)(x)(y, z) = D^2(g(f(x)))(Df(x)(z), Df(x)(y)) + Dg(f(x))D^2f(x)(y, z).$$

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x^2 - y^2, 2xy)$. Show that this function is locally invertible at all points $(x, y) \neq (0, 0)$. If we set $(u, v) = f(x, y)$ (that is $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$), then compute $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.
Solution: We compute the derivative of f to be

$$Df(x, y) = \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix}.$$

So the determinant of this matrix is $2x^2 + 2y^2$. Thus the linear map is invertible as long as $(x, y) \neq (0, 0)$. So the implicit function theorem says that the map f is locally invertible for $(x, y) \neq (0, 0)$.
We know $Df^{-1}(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ and by the implicit function theorem it is $Df^{-1}(u, v) = (Df)^{-1}(x, y)$ where $f(x, y) = (u, v)$. So

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{1}{2x^2 + 2y^2} \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

From this we see that $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v} = \frac{2x}{2x^2+2y^2}$ and $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u} = \frac{-2y}{2x^2+2y^2}$.

10. In an earlier homework assignment you showed that

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies $f(0) = 0, f'(0) \neq 0$ but that f is not locally invertible near 0. How does this not contradict the inverse function theorem?
Solution: The derivative is not continuous at $(x, y) = (0, 0)$.

11. Given the system of equations

$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + 2x + 3z^2, \end{aligned}$$

can we always solve for x, y, z in terms of u, v, w near $(0, 0, 0)$? Explain.
Solution: Let $f(x, y, z) = (x + xyz, y + xy, z + 2x + 3z^2)$ so $(u, v, w) = f(x, y, z)$. So we are asking if we can invert f near $(0, 0, 0)$. To figure this out we compute the derivative

$$Df(x, y, z) = \begin{pmatrix} 1 + yz & xz & xy \\ y & 1 + x & 0 \\ 2 & 0 & 1 + 6z \end{pmatrix}.$$

At $(0, 0, 0)$ we have

$$Df(0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Since the determinant of this matrix is 1 we see it is invertible. From above we also see that Df is continuous near $(0, 0, 0)$. Thus the inverse function theorem says that we can invert f near $(0, 0, 0)$. Thus we can write $(x, y, z) = f^{-1}(u, v, w)$ near $(0, 0, 0)$.