Math 4431 - Fall 2009
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 5, 7, 9 and 12.* **Due: September 29**

1) Given a collection of topological spaces $\{X_\alpha\}_{\alpha\in J}$ show that the product topology on $\prod_{\alpha\in J} X_\alpha$ is the smallest topology for which each of the projection maps is continuous.

2) Let D^2 be the unit disk in $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$. Show that the upper hemisphere of S^2 is homeomorphic to D^2 and similarly for the lower hemisphere. Use this to show that S^2 is homeomorphic to $D^2 \cup_g D^2$ for some homomorphism $g : S^1 \to S^1$ where $S^1 = \partial D^2$. Determine g.

3) Let D^2 be the unit disk in $\mathbb{R}^2$. Let $\mathcal{D}$ be a decomposition of $\mathbb{R}^2$ whose only non-trivial set is D^2. Show $\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$.

4) Let $X = [0,1], A = \{0,1\} \subset X$ and $Y = [4,5]$. Define the map $f : A \to Y$ by $f(0) = 4$ and $f(1) = 5$. Show that $X \cup_f Y$ is homeomorphic to S^1.

5) Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$. Show that the decomposition space of X defined as

$$\mathcal{D} = \{S_r | r > 0\},$$

where $S_r = \{(x,y)|x^2 + y^2 = r^2\}$, is homeomorphic to $\mathbb{R}$.

6) Show that S^1 is not homeomorphic to $[0,1]$.
Hint: Think connectedness.

7) Show that if U is an open connected subset of $\mathbb{R}^2$, then it is path connected. Hint: Fix an $x_0 \in U$ and show that the set of points in U that can be joined to x_0 by a path is both open and closed in U.

8) Show that a closed subspace of a normal space is normal.

9) Show that a connected normal space having more than one point is uncountable. (Hint: use Urysohn's Lemma)

10) Show that the any subspace of a separable metric space is separable.

11) The Tietze extension theorem is

Theorem 1 *Let X be a normal space and A a closed subspace of X. Then any continuous function $f : A \to [0,1]$ can be extended to a continuous function $f : X \to [0,1]$.*

Show that Urysohn's Lemma follows from the Tietze extension theorem. (Remark: It is also true but harder that the Tietze extension theorem follows from Urysohn's Lemma. See the book for a proof.)

12) Show the continuous image of a separable space is separable.