Math 4431 - Fall 2009
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 7, 8 and 9.* **Due: October 15**

1) Is there a continuous surjective map $f : [0,1] \to \mathbb{R}^2$? Why or why not.

2) Show there is a continuous surjective map $f : \mathbb{R} \to \mathbb{R}^2$.

3) If X is a Peano space of more than one point and Y is any Peano space, then show there is a continuous surjective map $f : X \to Y$.
HINT: First find a continuous map from X onto $[0,1]$.

4) Let (X,d) be a metric space. A function $f : X \to X$ is *contracting* if there is some constant $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x,y)$, for all $x, y \in X$. Prove that if X is complete and f is contracting then f is continuous and f has exactly one fixed point. (A fixed point of f is a point $x \in X$ such that $f(x) = x$.)
HINT: Fix x_1 in X and consider the sequence $x_n = f(x_{n-1})$.

5) This is a multi part problem that will use the last problem to prove the existence of solutions to ODEs. That is if we are given a function $f(x,y)$ and a point (x_0, y_0) then any function ϕ that satisfies

$$\phi'(x) = f(x, \phi(x)), \quad \phi(x_0) = y_0$$

is said to solve the *initial value problem*; $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$. In this problem you will prove the following theorem.

Theorem 1 *Assume that $f(x,y)$ is continuous in x and there is some K such that $|f(x,y) - f(x,y')| \leq K|y-y'|$ for all x, y and y'. (Notice that this is true if f is differentiable with respect to y.) Fix (x_0, y_0). Then there is a unique solution to the initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ defined on some interval $I_\delta = [x_0 - \delta, x_0 + \delta]$ about x_0.*

a) Show that $\phi(x)$ solves the initial value problem on I_δ if and only if

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt.$$

for all $x \in I_\delta$.
b) Fix some large closed ball B around (x_0, y_0) and let M be a constant such that $|f(x,y)| \leq M$ for all $(x,y) \in B$. Choose some $\delta > 0$ such that $R = \{(x,y) | x \in [x_0 - \delta, x_0 + \delta] \text{ and } y \in [y_0 - M\delta, y_0 + M\delta]\}$ is contained in B and $\delta K < 1$ (where K is from the statement of the theorem). Set $X = \{\phi : I_\delta \to \mathbb{R} | \phi(x_0) = y_0 \text{ and } |\phi(x) - y_0| \leq M\delta\}$. Show that X is a closed subset of the set of continuous functions $I_\delta \to \mathbb{R}$ with the topology given by the sup-metric. Conclude that X is a complete metric space.
c) Define $T : X \to X$ by $T(\phi)(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$. Prove that $T(\phi) \in X$ if $\phi \in X$ and show that ϕ is a solution to the initial value problem on I_δ if and only if $T(\phi) = \phi$.
d) Show that $|T(\phi_1)(x) - T(\phi_2)(x)| \leq K\delta \max_{x \in I_\delta} |\phi_1(x) - \phi_2(x)|$. In particular that T is a contraction on the complete metric space X. Conclude the proof of the theorem using problem 4) above.

A space X is called *Lindelöf* if every open cover X has a countable subcover.

6) A regular, Lindelöf space is normal.

7) Let X be a second countable space, then X is Lindelöf.

8) For a metric space X the following are equivalent: (a) X is second countable, (b) X is Lindelöf and (c) X is separable.

9) Suppose X is a compact Hausdorff space and $f : X \to Y$ is a quotient map. Show the following are equivalent: (a) Y is Hausdorff, (b) f takes closed sets in X to closed sets in Y and (c) the set $\{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\}$ is closed in $X \times X$.