Math 4431 - Fall 2009
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 5, 7 and 10.* **Due: October 29**

1) Show a compact metric space is second countable and separable.
Hint: Think totally bounded.
This actually follows immediately from Problem 8) on the previous homework assignment, but do not use this. Please come up with an argument using total boundedness of a compact metric space.

2) Let (X, d) be a compact metric space. If every bounded closed subset of $C(X, \mathbb{R})$ is compact, then show that X consists of a finite number of points.
Hint: In every infinite metric space there is an infinite sequence such that no point of the sequence is a limit point of the sequence (you need to prove this). Now think about the Tietze extension theorem.

3) Let X be a compact space. Show that any metric on X that induces the topology on X is complete.

4) Show that if X is a non-compact metrizable space, then there is a metric on X inducing the given topology that is not complete.

5) Let X be a compact metric space and Y a Hausdorff space. If there is a surjective continuous map $f : X \to Y$ then Y is also a compact metric space.
Hint: Show that for all $y \in Y$ and open set U in X such that $f^{-1}(y) \subset U$ there is an open set V in Y such that $f^{-1}(y) \subset f^{-1}(V) \subset U$. Now try to find an countable basis for Y.

6) Show any regular connected space has uncountably many points.
Hint: Show that a countable regular space is not connected. It might be good to know that a regular Lindelöf space is normal (this is problem 6 on the last homework, while you should try to prove this you don't have to when you write up the solution to this problem.

7) Let A be an open subset of the complete metric space (X, d). Show that there is some metric on A inducing the subspace topology on A that is complete.
Hint: There is a continuous function $f : A \to \mathbb{R}$ defined by $f(x) = \frac{1}{d(x, X-A)}$ and the map $F : A \to X \times R$ defined by $F(x) = (x, f(x))$.

A subset A of a space X is called G_δ if there is a countable collection $\{U_i\}_{i=1}^\infty$ of open sets such that $A = \cap_{i=1}^\infty U_i$.

8) Show that a set A in a complete metric space (X, d) is a complete metric space (possibly with a new metric inducing its subspace topology) if it is a G_δ set in X. Conclude that the topology on the irrational numbers as a subset of $\mathbb{R}$ can be given a complete metric.
Hint: Consider a function $f_i(x)$ as in problem 7) for each U_i. Show any sequence $\{x_n\}$ in A that is Cauchy in d and for which $\{f_i(x_n)\}$ is a bounded sequence for each i converges to a point in A. Now consider $d'(x, y) = d(x, y) + \sum_{i=1}^\infty \frac{1}{2^i} \min\{1, |f_i(x) - f_i(y)|\}$.

9) A closed set in a metric space is G_δ.

10) Let X be a compact Hausdorff space and $\mathcal{A}$ be a closed sub-algebra of $C(X, \mathbb{R})$ which separates points but does not contain the identity function, then show there is a unique point $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X, \mathbb{R}) \text{ such that } f(x_0) = 0\}$.