Math 4432 - Spring 2019
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 9, 10, 11, 12, 14, 15, and 16.* **Due: February 8**

1. Find all the topologies on the set $\{a, b, c\}$. Which are homeomorphic.

   Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The **subspace topology** on A is the topology on A defined by $\mathcal{T}_A = \{U \cap A | U \in \mathcal{T}\}$.

2. Show $\mathcal{T}_A$ is a topology on A.

   This is not a homework problem but the following fact is useful and I encourage you to think about why it is true.
   Fact: If $\mathcal{B}$ is a basis for $\mathcal{T}$ show that $\mathcal{B}_A = \{U \cap A | U \in \mathcal{C}\}$ is a basis for $\mathcal{B}_A$.

3. Let $\mathbb{R}$ be the x-axis in $\mathbb{R}^2$. Show the subspace topology on $\mathbb{R}$ is the same as the standard topology on $\mathbb{R}$ defined in class.

   Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two topological spaces. Set $\mathcal{B} = \{U \times V | U \in \mathcal{T} \text{ and } V \in \mathcal{T}'\}$.

4. Show $\mathcal{B}$ is a basis for a topology on $X \times Y$. This is called the **product topology**.

5. Show the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the same as the standard topology on $\mathbb{R}^2$.

6. The product of two Hausdorff spaces is Hausdorff.

7. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps then show that $g \circ f : X \to Z$ is a continuous map.

8. Show the projection onto either factor of a product space is continuous. That is show the maps $X \times Y \to X : (x, y) \mapsto x$ and $X \times Y \to Y : (x, y) \mapsto y$ are continuous.

9. Given a map $f : Z \to X \times Y$ you can always think of it as defined by $f(z) = (g(z), h(z))$ where $g : Z \to X$ and $h : Z \to Y$. Show that f is continuous if and only if g and h are both continuous.

10. Finite sets in a Hausdorff space are closed.
    Hint: First prove that sets with one point in them are closed.

    A topological space X is called $2^{\textbf{nd}}$ **countable** if it has a countable basis.

    A set A in a topological space X is said to be **dense** in X if $\overline{A} = X$.

    A topological space X is said to be **separable** if it has a countable dense subset.

11. Show a set A in X is dense if and only if every non-empty set in a basis for the topology of X contains a point of A.

12. Show a 2^{nd} countable space X is 1^{st} countable and separable.

13. Show that if U is an open connected subset of $\mathbb{R}^2$, then it is path connected. Hint: Fix an $x_0 \in U$ and show that the set of points in U that can be joined to x_0 by a path is both open and closed in U.

14. Show that S^1 is not homeomorphic to $[0, 1]$.

A collection of sets $\mathcal{C} = \{C_\alpha\}_{\alpha \in I}$ has the **finite intersection property** if for every finite sub-collection $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ of $\mathcal{C}$ the intersection $\cap_{i=1}^n C_{\alpha_i}$ is non-empty.

15. Show a space X is compact if and only if every collection of closed sets $\{C_\alpha\}_{\alpha \in I}$ having the finite intersection property has $\cap_{\alpha \in I} C_\alpha \neq \emptyset$.
Hint: Think about the complements of the C_α's.

16. Is there a continuous surjective map from $[0, 1]$ to $\mathbb{R}^2$? Prove your answer.