VI. Seifert - Van Kampen Theorem

A. Free Products with amalgamation

we want to build groups from other groups

given A and B

the free product of A and B is the set $A*B$ of all sequences $x = (x_1\ x_2 \cdots x_m)$ some m

where $x_i \in A$ or B

$x_i \neq e$ (identity in either group)

x_i, x_{i+1} from different groups

we call x a word in the letters $A \cup B$ of length m

let e = empty word

define multiplication by

$$(x_1, \dots x_m)(y_1, \dots y_n) = \begin{cases} (x_1, \dots, x_m, y_1, \dots, y_n) & \text{if } x_n, y_1 \text{ in different factors} \\ (x_1, \dots, x_{m-1}, x_m y_1, y_2, \dots, y_n) & \text{if } x_n, y_1 \text{ in same factor and } x_n y_1 \neq e \\ (x_1, \dots, x_{m-1})(y_2, \dots y_n) & \text{if } x_n, y_1 \text{ in same factor and } x_n y_1 = e \end{cases}$$

by induction now

note: $e \cdot x = x \cdot e = x \quad \forall x$

$x^{-1} = (x_m^{-1}, \dots, x_1^{-1})$

exercise: check associativity (induct on length of y)

Nice property for free products:

let $i: A \to A*B$, $j: B \to A*B$ be the obvious inclusions

given any homomorphisms $\phi: A \to H$, $\psi: B \to H$, H any group

$\exists!$ homomorphism $f: A*B \to H$ such that

$$f \circ i = \phi \quad \text{and} \quad f \circ j = \psi$$

just apply ϕ, ψ to letters in a word

Pictorally

$$\begin{tikzcd} & A \arrow[d, "i"] \arrow[ddr, bend left, "\phi"] & \\ B \arrow[r, "j"'] \arrow[drr, bend right, "\psi"'] & A*B \arrow[dr, dashed, "f"] & \\ & & H \end{tikzcd}$$

exercise: Show if D is another group with the above property then $D \cong A*B$

(the above property is called a "universal property")

examples:

1) Recall $\mathbb{Z}$ is the free group on one generator, say g

$$\mathbb{Z} = \langle g \rangle = F_1$$

means free group on one generator

all words in g and g^{-1}

(i.e. all powers of g)

recall F_n is the free group on n generators

e.g. F_2 = all reduced words in $g_1, g_2, g_1^{-1}, g_2^{-1}$

Claim: $F_2 \cong \mathbb{Z} * \mathbb{Z}$

indeed $\underset{\langle g \rangle}{\mathbb{Z}} * \underset{\langle h \rangle}{\mathbb{Z}} = (\text{word in } g, g^{-1}) \cdot (\text{word in } h, h^{-1}) \cdot \ldots$

let $X = \{g_1, g_2\}$ generate F_2

set $f : X \to \mathbb{Z} * \mathbb{Z} : \begin{cases} g_1 \longmapsto g \\ g_2 \longmapsto h \end{cases}$

since F_2 a free group on X, $\exists!$ homeomorphism

$$\tilde{f} : F_2 \to \mathbb{Z} * \mathbb{Z}$$

extending f

also $\mathbb{Z} = \langle g \rangle$ free so $\exists!$ homeomorphism

$$\phi : \mathbb{Z} \to F_2$$

defined by $g \mapsto g_1$

similarly for $\mathbb{Z} = \langle h \rangle$ we have

$$\psi : \mathbb{Z} \to F_2$$

defined by $h \mapsto g_2$

by property of free products above $\exists !$ homomor.

$$h : \mathbb{Z} * \mathbb{Z} \longrightarrow F_2$$

that agrees with ϕ and ψ on $\langle g \rangle, \langle h \rangle$, resp.

note: $\tilde{f} \circ h : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ is the identity

$h \circ \tilde{f} : F_2 \longrightarrow F_2$ " "

so $\tilde{f}$ and h are isomorphisms ✓

exercise: more generally $F_n \cong \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$

$\cong F_k * F_{n-k} \quad 0 \leq k \leq n$

2) recall $\mathbb{Z}_2 =$ integers modulo 2

$$\mathbb{Z}_2 * \mathbb{Z}_2 = \{ \underbrace{g_1 g_2 \cdots g_1 g_2}_{k \text{ times}}, \underbrace{g_1 g_2 \cdots g_1 g_2}_{k \text{ times}} g_1, \underbrace{g_2 g_1 \cdots g_2 g_1}_{k \text{ times}}, \underbrace{g_2 g_1 \cdots g_2 g_1}_{k \text{ times}} g_2, e \}_{k=0}^{\infty}$$

exercise: check this

3) Recall a group presentation of G is an isomorphism from G to $\langle g_1, \ldots g_n \mid r_1, \ldots r_m \rangle$ where $\langle g_1 \ldots g_n \mid r_1 \ldots r_m \rangle$ is the group $F_n / \langle r_1 \ldots r_m \rangle$ where F_n is the free group gen. by $g_1, \ldots, g_n$ and $\langle r_1 \ldots r_m \rangle$ is the smallest normal subgroup of F_n containing the r_i

e.g. $\mathbb{Z}_n \cong \langle g \mid g^n \rangle$

if $\langle g_1 \ldots g_n \mid r_1 \ldots r_m \rangle$ and $\langle g_1' \ldots g_{n'}' \mid r_1' \ldots r_{m'}' \rangle$ are presentations of G_1 and G_2, respectively then $G_1 * G_2$ has presentation

$$\langle g_1 \ldots g_n, g_1' \ldots g_{n'}' \mid r_1 \ldots r_m, r_1' \ldots r_{m'}' \rangle$$

exercise: prove this

given groups G_1, G_2, and K and homomorphisms

$$\psi_1 : K \to G_1 \quad \text{and}$$
$$\psi_2 : K \to G_2$$

then the free product with amalgamation is

$$G_1 *_K G_2 = G_1 * G_2 / \langle \psi_1(k)\psi_2(k)^{-1} \rangle_{k \in K}$$

where $\langle \psi_1(k)\psi_2(k) \rangle_{k \in K}$ is the smallest normal subgroup of $G_1 * G_2$ containing the set $\{\psi_1(k)\psi_2(k)^{-1}\}_{k \in K}$

the idea here is that we have all words in the elements of G_1 and G_2 but if we see $\psi_1(k)$ in a word we can replace it with $\psi_2(k)$ (and vice versa)

e.g.
$$\begin{aligned} \cdots \psi_2(k) \cdots &= \cdots \psi_1(k)\psi_2(k)^{-1})\psi_2(k) \cdots \\ &= \cdots \psi_1(k)\,(\underbrace{\psi_2(k)^{-1}\psi_2(k)}_{e}) \cdots \\ &= \cdots \psi_1(k) \cdots \end{aligned}$$

nice property of free products with amalgamation:

the inclusion maps $i_j : G_j \to G_1 * G_2$ induce maps

$$\bar{i}_j : G_j \longrightarrow G_1 *_K G_2$$ (by composing with quotient map)

given any homomorphisms

$$\phi_1 : G_1 \to H$$
$$\phi_2 : G_2 \to H$$
any group

such that $\phi_1 \circ \psi_1(k) = \phi_2 \circ \psi_2(k) \quad \forall k \in K$

there exists a unique homomorphism

$$\phi : G_1 *_K G_2 \longrightarrow H$$

such that

$$\phi \circ \bar{\imath}_1 = \phi_1 \quad \text{and} \quad \phi \circ \bar{\imath}_2 = \phi_2$$

Pictorally

$$\begin{tikzcd}
K \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] \arrow[dr, phantom, "\circ"] & G_1 \arrow[d, "\bar{\imath}_1"] \arrow[ddr, bend left, "\phi_1"] & \\
G_2 \arrow[r, "\bar{\imath}_2"'] \arrow[drr, bend right, "\phi_2"'] & G_1 *_K G_2 \arrow[dr, dashed, "\phi"] & \\
 & & H
\end{tikzcd}$$

exercise: Prove this

In terms of presentations

if $G_1 \cong \langle g_1 \dots g_n \mid r_1 \dots r_m \rangle$

$G_2 \cong \langle g_1' \dots g_{n'}' \mid r_1 \dots r_{m'}' \rangle$

$K \cong \langle h_1 \dots h_\ell \mid r_1'' \dots r_{m''}'' \rangle$

then

$$G_1 *_K G_2 \cong \langle g_1 \dots g_n, g_1' \dots g_{n'}' \mid r_1 \dots r_m, r_1' \dots r_{m'}', \psi_1(h_1)\psi_2(h_1)^{-1} \dots \psi_1(h_\ell)\psi_2(h_\ell)^{-1} \rangle$$

exercise: Prove this

B Seifert-Van Kampen Theorem

So far we have only been able to compute π_1 of spaces homotopy equivalent to a point or S^1

With the following theorem we can do much more!

Th$^{\underline{m}}$ 1 (Seifert-Van Kampen):

let X be a topological space with base point x_0

suppose $X = A \cup B$ where

A and B are path connected open sets,

$A \cap B$ is path connected, and

$x_0 \in A \cap B$

let $\psi_A : \pi_1(A \cap B, x_0) \longrightarrow \pi_1(A, x_0)$ and

$\psi_B : \pi_1(A \cap B, x_0) \longrightarrow \pi_1(B, x_0)$

be the homomorphisms induced by the inclusion maps

$$A \cap B \begin{matrix} \subset A \\ \subset B \end{matrix}$$

Then

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$

before we sketch a proof, let's look at a few examples

examples:

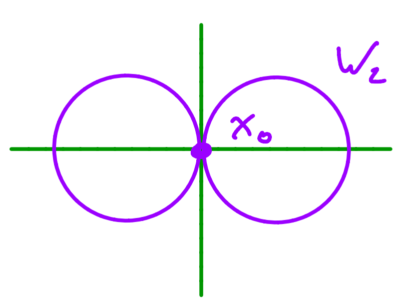
1) let W_2 = "wedge of 2 circles"

i.e. take $x_1 \in S^1$ be a fixed point on S^1

$x_2 \in S^1$ be a fixed point on a copy of S^1

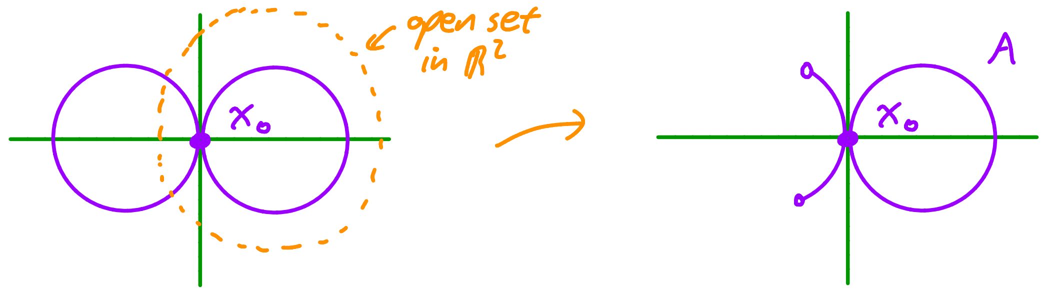
then $W_2 = S^1 \cup S^1 / \underbrace{\{x_1 \sim x_2\}}_{x_0}$

we can think of W_2 as a subset of $\mathbb{R}^2$

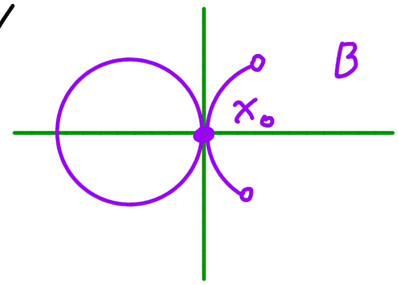


i.e. $W_2 = \{(x,y) \mid (x-1)^2 + y^2 = 1\}$
$\cup \{(x,y) \mid (x+1)^2 + y^2 = 1\}$

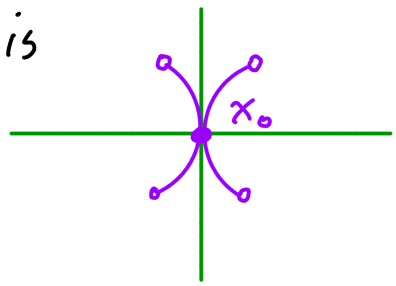
we need path connected open sets A and B



similarly



so $A \cap B$ is



pick x_0 to be the "wedge point" = origin

exercise:

1) A and B are homotopy equivalent to S^1

Hint: Show A and B are homeomorphic to

$S^1 \vee (-1,1) / \{x_1 \sim x_2\}$ where $x_1 \in S^1$, $x_2 = 0 \in (-1,1)$

then use homotopy equivalence of $(-1,1)$ to x_0 to give the desired homotopy equivalence

2) $A \cap B$ is homotopy equivalent to $\{x_0\}$

so $\pi_1(A, x_0) \cong \mathbb{Z} \cong \langle g_1 \mid \ \rangle$

$\pi_1(B, x_0) \cong \mathbb{Z} \cong \langle g_2 \mid \ \rangle$

$\pi_1(A \cap B, x_0) = \{e\}$

and $\psi_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0)$

$$e \longmapsto e$$

and similarly for ψ_B

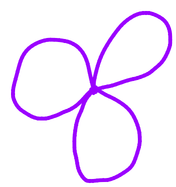
thus $\pi_1(W_2, x_0) \cong \mathbb{Z} *_{\{e\}} \mathbb{Z}$

$$\begin{aligned} &\cong \langle g_1, g_2 \mid \psi_A(e)\psi_B(e)^{-1} \rangle \\ &\cong \langle g_1, g_2 \mid ee^{-1} \rangle = \langle g_1, g_2 \mid e \rangle \\ &\cong \langle g_1, g_2 \mid \rangle \cong F_2 \end{aligned}$$

why!

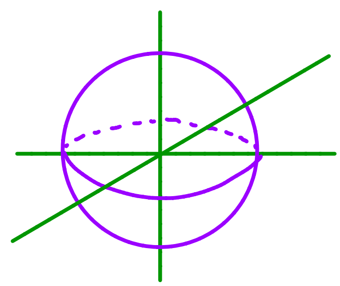
so $\pi_1(W_2, x_0)$ is the free group on 2 generators

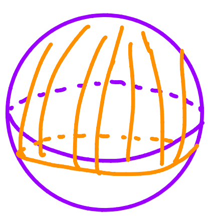
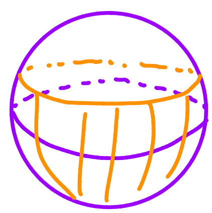
exercise: if W_n = wedge of n circles then $\pi_1(W_n, x_0) \cong F_n$

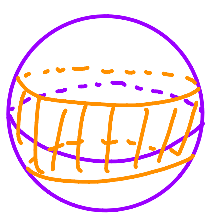


W_3

2) consider $S^2 \subset \mathbb{R}^3$



let $A =$  and $B =$ 

so $A \cap B$ is  pick x_0 on the equator

we know A and B are disks so each is $\simeq \{x_0\}$

$A \cap B \cong$ annulus $\simeq S^1$

so $\pi_1(A, x_0) \cong \{e\} \cong \pi_1(B, x_0) \cong \langle \mid \rangle$

$\pi_1(A \cap B, x_0) \cong \mathbb{Z} \cong \langle g \mid \rangle$

$$\psi_A : \pi_1(A\cap B, x_0) \longrightarrow \pi_1(A, x_0) : g^n \longmapsto e \quad \forall n$$

similarly for ψ_B

no generators

so $\pi_1(S^2, x_0) \cong \{e\} *_{\mathbb{Z}} \{e\} \cong \langle \ | \ \psi_A(g)\,\psi_B(g)^{-1} \rangle$

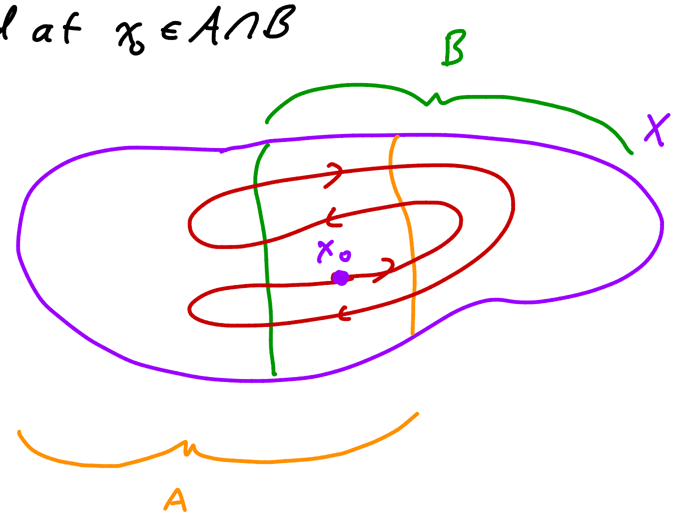
$\cong \{e\}$ trivial group

$\therefore \pi_1(S^2, x_0)$ is the trivial group

we will see more complicated examples later, but first

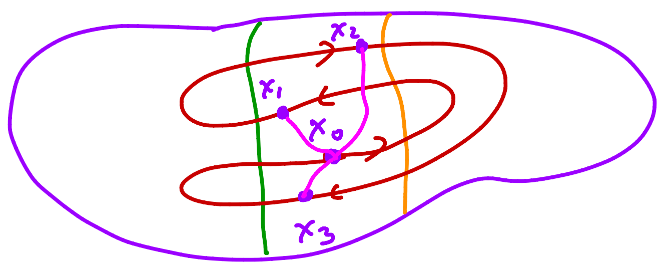
Idea of proof of Seifert-Van Kampen Theorem:

given a loop γ in X based at $x_0 \in A\cap B$

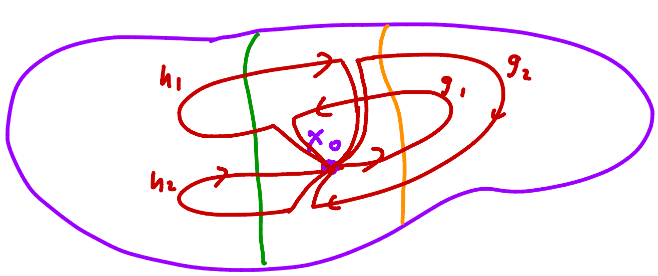


you can pick points $x_1, \ldots x_k$ in $\gamma \cap (A\cap B)$ s.t. arc x_i to x_{i+1} in A or B (use lebesgue number)

now use path connectedness of $A\cap B$ to choose arcs in $A\cap B$ connecting x_0 to x_i



now consider



this gives γ' homotopic to γ written as a product of elements from $\pi_1(A,x_0)$ and $\pi_1(B,x_0)$

$$g_1 h_1 g_2 h_2$$

the inclusion maps give

$$\phi_A : \pi_1(A,x_0) \longrightarrow \pi_1(X,x_0)$$
$$\phi_B : \pi_1(B,x_0) \longrightarrow \pi_1(X,x_0)$$

note for $h \in \pi_1(A \cap B, x_0)$ we see

$$\phi_A \circ \psi_A(h) = i_*(h)$$
$$\phi_B \circ \psi_B(h) = i_*(h)$$

where $i : A \cap B \to X$ is inclusion

so the universal property for free products with amalgamation says we get a homomorphism

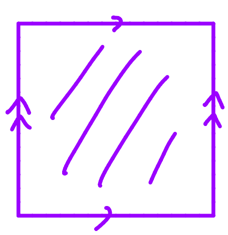
$$\phi : \pi_1(A,x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B,x_0) \longrightarrow \pi_1(X,x_0)$$

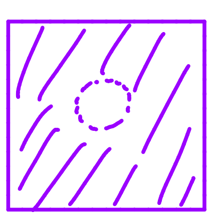
the above argument says ϕ is onto

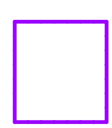
we are left to see ϕ is injective (see any book on algebraic topology for this)

C. Fundamental Group, Surfaces, and 1^{st} Homology

let's compute $\pi_1(T^2, x_0)$

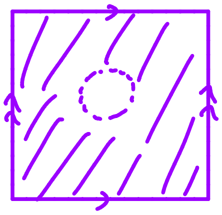
$T^2 \cong$ 

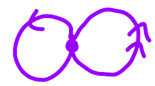
let $\tilde{A} =$ 

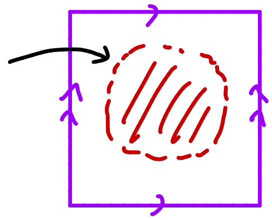
exercise: $\tilde{A} \simeq$  $\cong S^1$

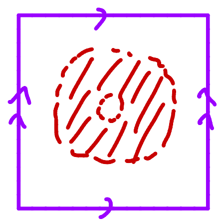
homotopy equivalent

homeomorphic

let $A =$  so $\tilde{A} \simeq S^1$ gives

$A \simeq$  wedge of 2 circles !

let B  so $B \simeq$ point

note $A \cap B$ is  so $A \cap B \simeq S^1$

and $T^2 = A \cup B$ pick $x_0 \in A \cap B$

now $\pi_1(A, x_0) \cong F_2 \cong \langle g_1, g_2 \mid \ \rangle$

$\pi_1(B, x_0) = \{e\}$

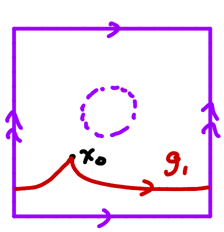
$\pi_1(A \cap B, x_0) \cong \mathbb{Z} \cong \langle h \mid \ \rangle$

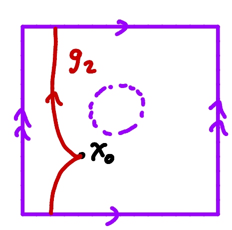
note $\psi_B : \pi_1(A \cap B, x_0) \longrightarrow \pi_1(B, x_0)$

$h^n \longmapsto e \qquad \forall n$

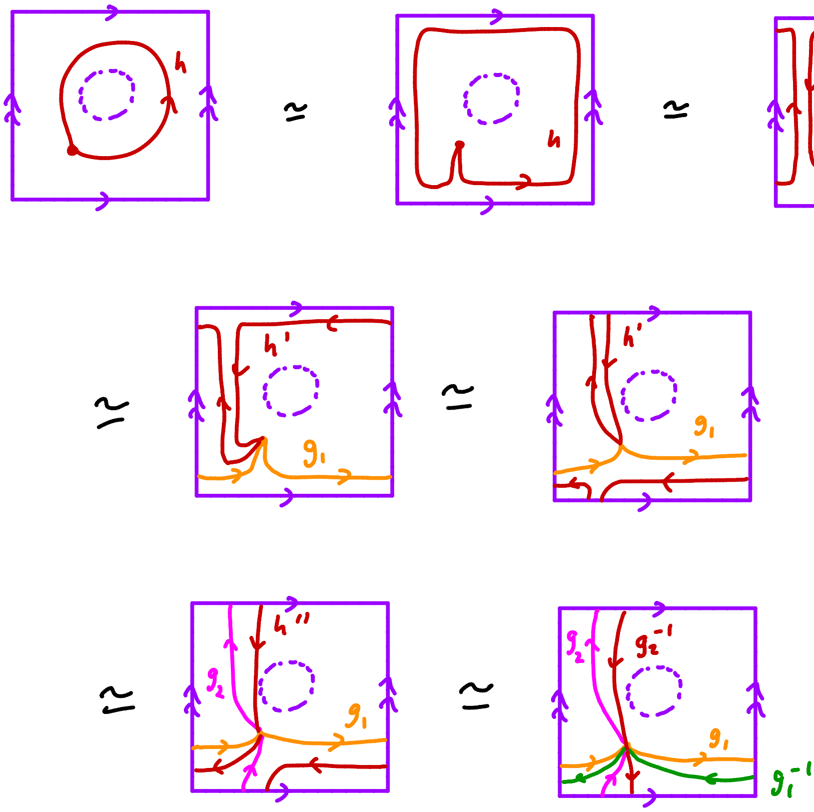
for $\psi_A : \pi_1(A \cap B, x_0) \longrightarrow \pi_1(A, x_0)$

we claim $\psi_A(h) = g_1 g_2 g_1^{-1} g_2^{-1}$

indeed note 



so



$$\psi_A(h) \simeq g_1 g_2 g_1^{-1} g_2^{-1}$$

so we have

$$\pi_1(T^2, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$
$$\cong \langle g_1 g_2 | \ \rangle *_{\langle h | \ \rangle} \langle 1 \rangle$$
$$\cong \langle g_1, g_2 \mid \psi_A(h)\, \psi_B(h)^{-1} \rangle$$
$$\cong \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1} \rangle$$

$g_1 g_2 g_1^{-1} g_2^{-1}$ is called the <u>commutator</u> of g_1 and g_2 and is denoted $[g_1, g_2]$

the relation says g_1 and g_2 commute

$$g_1 g_2 g_1^{-1} g_2^{-1} = e$$
$$g_1 g_2 g_1^{-1} g_2^{-1} g_2 = e\, g_2$$
$$g_1 g_2 g_1^{-1} = g_2$$
$$g_1 g_2 g_1^{-1} g_1 = g_2 g_1$$
$$g_1 g_2 = g_2 g_1$$

we saw earlier that this is a presentation for $\mathbb{Z} \oplus \mathbb{Z}$

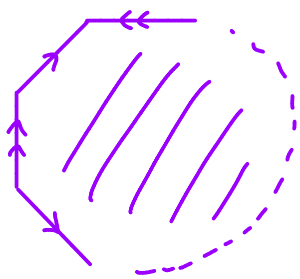
so $\pi_1(T^2, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$

since $\pi_1(S^2, x_0) = \{e\}$ we see S^2 and T^2 are not homeomorphic

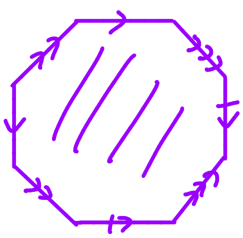
we already knew this, but now we see they are also not even homotopy equivalent!

now if Σ_g is a surface of genus g, then recall

$$\Sigma_g = 4g\text{-gon with edges identified}$$



eg Σ_2 is



exercise: Show

$$\pi_1(\Sigma_g, x_0) \cong \langle g_1, \dots, g_{2g} \mid [g_1, g_2][g_3, g_4] \cdots [g_{2g-1}, g_{2g}] \rangle$$

are these groups different?

If G is any group, its commutator subgroup $[G,G]$ is the smallest normal subgroup of G containing $\{ghg^{-1}h^{-1}\}_{g,h \in G}$

the abelionization of G is $G/[G,G]$

exercise:

1) Show $G/[G,G]$ is abelian

2) if $G \cong \langle g_1 \dots g_n \mid r_1 \dots r_m \rangle$ then

$$G/[G,G] \cong \langle g_1 \dots g_n \mid r_1 \dots r_m \ [g_i, g_j] \ i,j = 1, \dots, n \rangle$$

3) if $G \cong H$, then $G/_{[G,G]} \cong H/_{[H,H]}$

if X is a path connected topological space and $x_0 \in X$ then the first homology group of X is

$$H_1(X) \cong \pi_1(X,x_0) / _{[\pi_1(X,x_0), \pi_1(X,x_0)]}$$

so $H_1(\Sigma_g) \cong \langle g_1 \dots g_{2g} \mid [g_1,g_2] \cdots [g_1, g_{2g}], [g_1,g_2], \dots \rangle$

note the first relation follows from all the other relations, so we can discard it

$H_1(\Sigma_g) \cong \langle g_1 \dots g_{2g} \mid [g_i, g_j] \quad i,j = 1, \dots 2g \rangle$

exercise: $\langle g_1 \dots g_{2g} \mid [g_i, g_j] \quad i,j = 1, \dots 2g \rangle \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{2g \text{ times}} \cong \oplus_{2g} \mathbb{Z}$

(i.e. Σ_g has "2g independent holes")

now is $\oplus_k \mathbb{Z} \ncong \oplus_\ell \mathbb{Z}$ if $k \neq \ell$?

recall $\oplus_k \mathbb{Z} \subseteq \mathbb{R}^k$ (set of integer points)

subset

and group operation in $\oplus_k \mathbb{Z}$ is just vector addition

$\mathbb{R}^k$ is spanned by $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

and these are all in $\oplus_k \mathbb{Z}$

a linear map on $\mathbb{R}^k$ is determined by what it does on a basis

so any homomorphism $\phi: \oplus_k \mathbb{Z} \to \oplus_\ell \mathbb{Z}$ will induce a linear map $\Phi: \mathbb{R}^k \to \mathbb{R}^\ell$

exercise: ϕ is a group isomorphism $\Rightarrow$ Φ is a vector space isomorphism

this implies $\oplus_k \mathbb{Z} \cong \oplus_\ell \mathbb{Z} \Leftrightarrow k=\ell$

$$\therefore \; H_1(\Sigma_g) \cong H_1(\Sigma_h) \Leftrightarrow g=h$$

Thm 2:

$$\boxed{\begin{aligned} \Sigma_g \cong \Sigma_h &\Leftrightarrow \Sigma_g \simeq \Sigma_h \Leftrightarrow g=h \\ &\Leftrightarrow \chi(\Sigma_g) = \chi(\Sigma_h) \\ &\Leftrightarrow H_1(\Sigma_g) = H_1(\Sigma_h) \Leftrightarrow \pi_1(\Sigma_g, x_0) \cong \pi_1(\Sigma_g, y_0) \end{aligned}}$$

exercise:

1) Show the fundamental group of N_n is

$$\pi_1(N_n, x_0) \cong \langle g_1 \cdots g_n \mid g_1^2 \cdots g_n^2 \rangle$$

2) Show the fundamental group of $\Sigma_{g,k}$ for $k>0$ is

$$\pi_1(\Sigma_{g,k}, x_0) \cong F_{2g+k-1}$$

and for $N_{n,k}$ for $k>0$ is

$$\pi_1(N_{n,k}, x_0) \cong F_{n+k-1}$$

D. Groups and Topology

We will now see how to build topological spaces realizing a given group as its fundamental group and how to realize group homomorphisms via continuous maps!

(i.e. turn algebra into topology!)

let D^n = unit disk in $\mathbb{R}^n$

$S^{n-1} = \partial D^n$

given a topological space Y and a continuous map

$$a : S^{n-1} \to Y$$

the space obtained from Y ba attaching an n-cell is

$$Y \cup_a D^n = Y \amalg D^n / \{x \sim a(x)\}_{x \in S^{n-1}}$$

of course $Y \cup_a D^n$ has the quotient topology

we can similarly attach many n-cells at one time

i.e. given $a = \coprod_\lambda a_\lambda \qquad a_\lambda : S^{n-1}_\lambda \to Y$

then $\quad Y \cup_a \coprod_\lambda D^n_\lambda$

an n-complex, or n-dimensional CW complex is defined

inductively by

a (-1)-complex is $\emptyset$

an n-complex is any space obtained by

attaching n-cells to an $(n-1)$-complex

an n-complex is finite if it has finitely i-cells for all

i between 0 an n

the k-skeleton of an n-complex X is the union of all

i-cells for $i \leq k$, it is denoted by $X^{(k)}$

(can define ∞-dimensional complexes as

$$X = \bigcup_{n=0}^{\infty} X_n$$ where X_n is an n-complex obtained from X_{n-1} by attaching n-cells

here U in X is open $\Leftrightarrow$ $U \cap X_n$ open $\forall n$)

this is called the weak topology on X

explains the W in CW

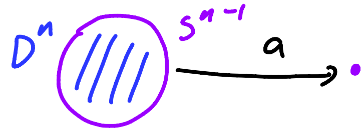
Fact: CW complexes are Hausdorff

examples:

1) any n-simplicial complex is an n-complex
2) S^n is an n-complex

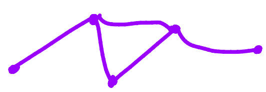
   0-skeleton is $\bullet$

   attach n-cell by $a: \partial D^n \to \{pt\}$
   constant map

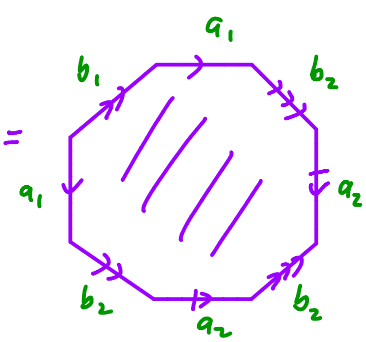


   i.e. S^n is D^n with the boundary collapsed to a point

3) 1-complexes are graphs
   (and graphs are 1-complexes)

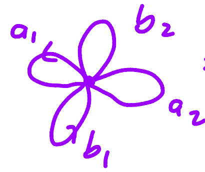


4) Compact surfaces without boundary is a 2-complex

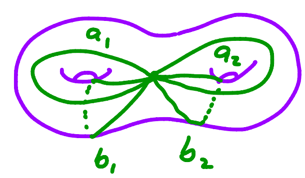
$P =$ 

$\Sigma_g = P/\sim$

1 0-cell | 2g 1-cells | 1 2-cell P itself



$= \partial P/\sim$



5) **Fact:** any (differentiable) manifold is homotopy equivalent to a CW-complex

lemma 3:

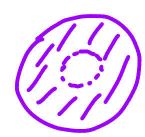
> let X be a topological space and
> $$a : \partial D^2 \longrightarrow X$$
> be a continuous map.
> let $1 \in \pi_1(\partial D^2, p_0) \cong \mathbb{Z}$ be a generator and
> $$r = a_*(1) \in \pi_1(X, x_0) \quad \text{where } x_0 = a(p_0)$$
> If $Y = X \cup_a D^2$, then
> $$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/\langle r \rangle$$
> smallest normal subgroup containing r

so "attaching a 2-cell" adds a relation to the fundamental group

exercise: Show that if Y is obtained from X by attaching an n-cell with $n \geq 3$, then $\pi_1(Y, x_0) \cong \pi_1(X, x_0)$

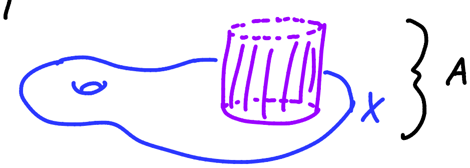
Proof: We use Seifert-Van Kampen Th^{m}

let $A = X \cup_a \underbrace{S^1 \times (1/2, 1]}_{\text{subset of } D^2}$



note: A is an open set in Y

exercise: $A \simeq X$

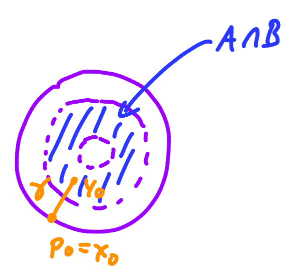


let B = disk of radius $2/3 \subset D^2$

so $A \cap B = S^1 \times (1/2, 2/3) \simeq S^1$

take $y_0 \in A \cap B$

$$\pi_1(A \cap B, y_0) \cong \mathbb{Z}$$
$$\pi_1(B, y_0) = \{e\}$$
$$\pi_1(A, y_0) \cong \pi_1(X, y_0) \cong \pi_1(X, x_0)$$

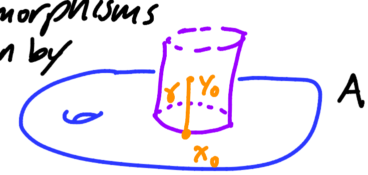


exercise:

$$\begin{array}{ccc} \pi_1(A \cap B, y_0) & \xrightarrow{i_*} & \pi_1(A, y_0) \\ \wr\wr & & \wr\wr \\ \pi_1(\partial D^2, p_0) & \xrightarrow{a_*} & \pi_1(X, x_0) \end{array}$$

i is inclusion

Hint: both isomorphisms given by γ



so $i_*(1) = a_*(1) = r$

so $\pi_1(Y, y_0) \cong \pi_1(A, y_0) *_{\pi_1(A\cap B, y_0)} \pi_1(B, y_0)$

$$\cong \pi_1(X, x_0) * \{e\} \Big/ \langle i_*(1)e^{-1}\rangle$$

$$\cong \pi_1(X, x_0)/\langle r\rangle$$

Thm 4: let G be a group. Then $\exists$ a topological space X (in fact a 2-complex) such that $\pi_1(X, x_0) \cong G$

Proof: we consider a group G with a finite presentation

$$\langle g_1 \dots g_n \mid r_1 \dots r_m\rangle$$

the general case is almost the same but need to be happy with infinite complexes

let W_n = wedge of n circles (so a 1-complex)

recall $\pi_1(W_n, x_0) \cong F_n \cong \langle g_1 \dots g_n \mid \ \rangle$

let $a_i : \partial D^2 \to W_n$ be a continuous map such that

$$(a_1)_* : \pi_1(\partial D^2, p_0) \longrightarrow \pi_1(W_n, x_0)$$
$$1 \longmapsto r_i$$

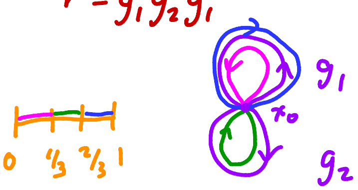
exercise: construct a_i

Hint: if $r_i = g_{j_1}^{\varepsilon_1} \cdots g_{j_k}^{\varepsilon_k}$ $\quad \varepsilon_i = \pm 1$

then define r_i on $[j/k, j+1/k]$ $\quad j = 0, \dots, k-1$

to map onto the loop in W_n corresponding to g_{j_i} agreeing with orientation or not depending on ε_i

e.g. $r = g_1 g_2 g_1^{-1}$



let $X = W_n \cup_{a_i} \left(\coprod_{i=1}^{m} D^2\right)$

lemma 3 $\Rightarrow \pi_1(X, x_0) \cong \langle g_1 \dots g_n \mid r_1 \dots r_m\rangle$

Th^m 5:

let G and H be any groups, and

$$\phi : G \to H$$

any homomorphism.

let X, Y be topological spaces such that

$$\pi_1(X, x_0) \cong G \quad \text{and} \quad \pi_1(Y, y_0) \cong H$$

If X is a 2-complex, then $\exists$ a continuous function $f : X \to Y$

such that $f_* = \phi$

Remark: Note this implies that any homomorphism between the fundamental groups of surfaces is induced by a continuous map!

Proof: Though not necessary we take X to be the 2-complex defined in Th^m 4

so $\pi_1(X, x_0) \cong G \cong \langle g_1 \dots g_n \mid r_1 \dots r_m \rangle$

let γ_i be any loop in Y based at y_0

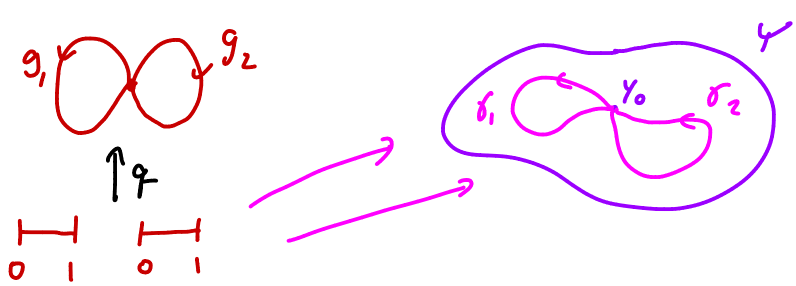
s.t. $[\gamma_i] = \phi(g_i) \in \pi_1(Y, y_0)$

i.e. $\gamma_i : [0,1] \to Y$ s.t. $\gamma_i(0) = \gamma_i(1) = y_0$

$[\gamma_i] = \phi(g_i)$

now $X = W_n \cup_{q_?} (\amalg D^2)$

define $f : W_n \to Y$ on the g_i loop by γ_i



recall $W_n = \hat{\coprod}_{i=1} [0,1]/\sim$ where all end points are identified

so on i^{th} $[0,1]$ define f to be γ_i
this decends to the quotient space

we want to extend f over each 2-cell in X

let D^2 be the 1^{st} 2-cell (same argt for others)

note $a_1(\partial D^2)$ is a loop in W_n representing the relation r_1

so $[a_1(\partial D^2)] = e \in \pi_1(X, x_0) \cong G$ (note $x_0 \in a_1(\partial D^2)$)

$\therefore \phi(\underbrace{[a_1(\partial D^2)]}_{\shortparallel}) = e$ in $\pi_1(Y, y_0)$

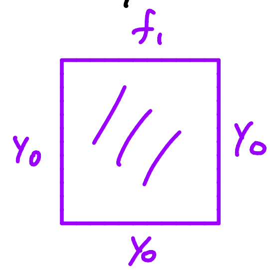
$f_1 : f \circ a_1 : S^1 \to Y$

think of map on S^1 as on $[0,1]$

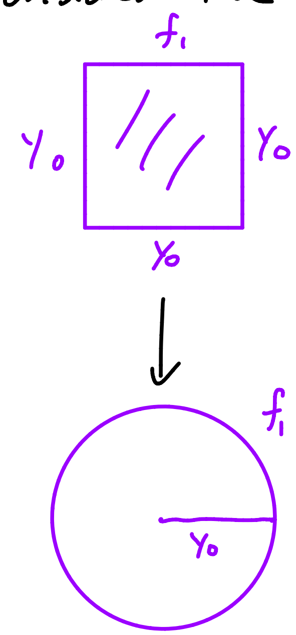
so $f_1 : [0,1] \longrightarrow Y$ is homotopic to the trivial loop

that is $\exists$ homotopy $H : \underset{s}{[0,1]} \times \underset{t}{[0,1]} \longrightarrow Y$

s.t.



consider the quotient map



$$\begin{array}{ccc} [0,1] \times [0,1] & \xrightarrow{H} & Y \\ \Big\downarrow & (s,t) & \\ & \downarrow & \\ D^2 & (t\cos 2\pi s, t \sin 2\pi s) & \end{array}$$

clearly H induces a map
$$\tilde{H}: D^2 \to Y$$
such that $\tilde{H}|_{\partial D^2} = f \circ a_1$

so use $\tilde{H}$ to extend f over the 1^{st} 2-cell

continuing we get $f: X \to Y$

by construction $f_* = \phi$ on the g_i so they are equal on all of G $\blacksquare$

Th$^{\underline{m}}$6:

let Σ, Σ' be compact surfaces without boundary
Σ' <u>not</u> homeomorphic to S^2 or P^2
Then $f_0, f_1 : \Sigma \to \Sigma'$ $(f_i(x_0) = y_0)$ are homotopic (base pt preserving)
$$\Longleftrightarrow$$
$$(f_0)_* = (f_1)_* : \pi_1(\Sigma, x_0) \to \pi_1(\Sigma', y_0)$$

"maps between (most) surfaces are determined (upto homotopy) by their action on π_1"

Remark:

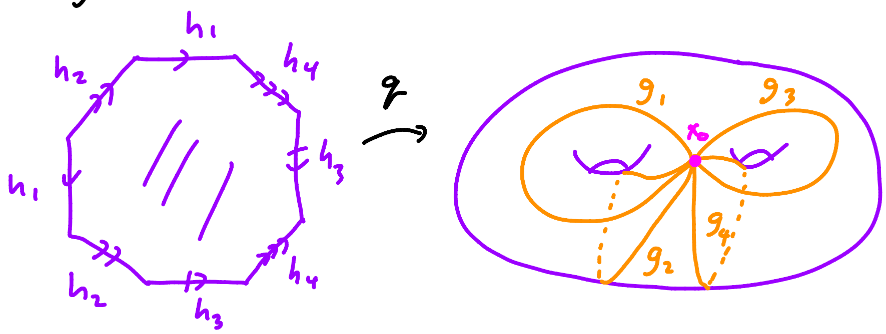
1) not true in higher dimensions
2) $\{\text{homotopy classes } S^2 \to S^2\} \longleftrightarrow \mathbb{Z}$

Proof: ($\Rightarrow$) exercise in Section II just after Th$^{\underline{m}}$3

($\Leftarrow$) need to define
$$H: \Sigma \times [0,1] \to \Sigma'$$
s.t. $H(x,0) = f_0(x)$
$H(x,1) = f_1(x)$

now let $g_1 \dots g_{2g}$ be generators of $\pi_1(\Sigma, x_0)$

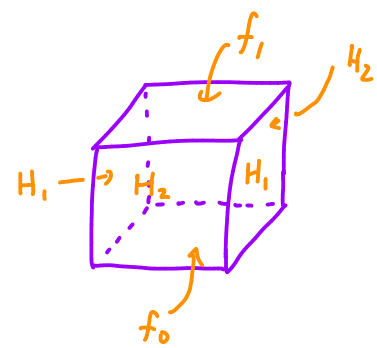
coming from



since $(f_0)_*(g_1) = (f_1)_*(g_2)$

we know $f_0 \circ g_2 \simeq f_1 \circ g_1$

let H_2 be this homotopy



define H on $g_1 \times [0,1]$ by H_i

on $\Sigma \times \{i\}$ by f_i

<u>note</u>: $(\Sigma \times [0,1]) \setminus (\bigcup_{g_2} \times [0,1])$

$= (4n\text{-gon}) \times [0,1] = B^3$

<u>Fact (we prove this later)</u>: any map $S^2 \to \Sigma'$ is homotopic to a constant map

(here is where we need $\Sigma' \neq S^2$ or P^2)

now $H|_{\partial B^3} : S^2 \to \Sigma'$

since the map is homotopically trivial we get a homotopy

$G : S^2 \times [0,1] \to \Sigma^2$

$G(p,0) = c$ $\qquad$ c some fixed point in Σ'

$G(p,1) = H(p)$

so G induces a map $\tilde{G} : \underbrace{S^2 \times [0,1] / S^2 \times \{0\}}_{B^3} \to \Sigma'$

use $\tilde{G}$ to extend H over rest of $\Sigma \times [0,1]$