Math 4432 - Spring 2021
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 3, 4, 6, 7, and 11.* **Due: February 12**

1. Show that if U is an open connected subset of $\mathbb{R}^2$, then it is path connected. Hint: Fix an $x_0 \in U$ and show that the set of points in U that can be joined to x_0 by a path is both open and closed in U.

2. Show that S^1 is not homeomorphic to $[0, 1]$.

   A collection of sets $\mathcal{C} = \{C_\alpha\}_{\alpha \in I}$ has the **finite intersection property** if for every finite sub-collection $\{C_{\alpha_1}, \dots, C_{\alpha_n}\}$ of $\mathcal{C}$ the intersection $\cap_{i=1}^n C_{\alpha_i}$ is non-empty.

3. Show a space X is compact if and only if every collection of closed sets $\{C_\alpha\}_{\alpha \in I}$ having the finite intersection property has $\cap_{\alpha \in I} C_\alpha \neq \emptyset$.
   Hint: Think about the complements of the C_α's.

4. Is there a continuous surjective map from $[0, 1]$ to $\mathbb{R}^2$? Prove your answer.

5. Let $X = [0, 1], A = \{0, 1\} \subset X$ and $Y = [4, 5]$. Define the map $f : A \to Y$ by $f(0) = 4$ and $f(1) = 5$. Show that $X \cup_f Y$ is homeomorphic to S^1.

6. Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$. Show that the decomposition space of X defined as

   $$\mathcal{D} = \{S_r | r > 0\},$$

   where $S_r = \{(x, y) | x^2 + y^2 = r^2\}$, is homeomorphic to $\mathbb{R}$.

7. Let $X = S^1 \times [0, 1]$ and consider the decomposition space

   $$\mathcal{D} = \{\{(x, t)\} : t \in (0, 1] \text{ and } x \in S^1\} \cup \{(x, 0) : x \in S^1\},$$

   that is the only non-trivial set in the decomposition is $S^1 \times \{0\}$. Prove that $\mathcal{D}$ is homeomorphic to D^2.
   Remark: You might also want to try to show that if S^1 is replaced by S^n then the analogous decomposition space is homeomorphic to D^{n+1}.

8. Let D^2 be the unit disk in $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$. Show that the upper hemisphere of S^2 is homeomorphic to D^2 and similarly for the lower hemisphere. Use this to show that S^2 is homeomorphic to $D^2 \cup_g D^2$ for some homomorphism $g : S^1 \to S^1$ where $S^1 = \partial D^2$. Determine g.

9. Let D^2 be the unit disk in $\mathbb{R}^2$. Let $\mathcal{D}$ be a decomposition of $\mathbb{R}^2$ whose only non-trivial set is D^2. Show $\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$.

10. Let D be a disk and I be an interval in ∂D. If Σ is a surface and $f : I \to \partial\Sigma$ is an embedding, then show the surface

    $$\Sigma \cup_f D$$

is homeomorphic to Σ.
Hint: It might be good to try to show that the space obtained from a disk and an annulus by gluing them along intervals in their boundary is homeomorphic to an annulus. You may assume, as discussed in class, that given any connected component B of $\partial\Sigma$ there is an open set U in Σ that contains B and is homeomorphic to $S^1 \times [0,1)$.

11. Show that for any connected surface Σ and points p and q in Σ there is a homeomorphism $h : \Sigma \to \Sigma$ such that $h(p) = q$.
Hint: Fix p and consider the set

$$S = \{q \in \Sigma \text{ such that there is a homeomorphism sending } p \text{ and } q\}.$$

Remark: Notice that from this problem you know for any non-empty surface there are an uncountable number if different homeomorphisms $\Sigma \to \Sigma$. Also, there are two extensions of this problem you might want to consider. 1) If M is a connected n-manifold and points p and q in M there is a homeomorphism $h : M \to M$ such that $h(p) = q$. 2) If M is a connected n-manifold, for $n > 1$, and $\{p_1, \ldots .p_k\}$ and $\{q_1, \ldots q_k\}$ are two collections of distinct points in M, then there is a homeomorphism $h : M \to M$ such that $h(p_i) = q_i$ for $i = 1, \ldots, k$. What happens for $n = 1$? Is there any similar statement that can be made?