Math 4432 - Spring 2021
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 5, 6, 10, 12, and 13.* **Due: March 12**

1. Recall the mapping class group of a space X, denoted by $\mathrm{Mod}(X)$, is the group of isotopy classes of homeomorphisms of X. Show $\mathrm{Mod}(X)$ is a group.

2. Prove $\mathrm{Mod}(S^1)$ is isomorphic to $\mathbb{Z}_2$.

3. Let G and H be groups and $f : G \to H$ a homomorphisms. Show that the image of f is a subgroup of H.

4. If $\phi : G \to H$ is a homomorphisms, then show that the image of ϕ is isomorphic to $G/\ker\phi$.

5. Show $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.
   HINT: Think of the orders of elements.

6. Show that $\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.

7. Show $\mathbb{Z} \oplus \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.
   HINT: Where would such an isomorphism send $1 \in \mathbb{Z}$?

8. Suppose a group G has a presentation $\langle x_1, \ldots, x_n | r_1, \ldots r_m\rangle$ where the relators are
   $$r_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{k_i}}^{s_{i_{k_i}}}$$
   for $i = 1, \ldots m$, and the s_i are ± 1. Then show that if H is any other group and $h_1, \ldots, h_n$ are any elements of H that satisfy
   $$h_{i_1}^{s_{i_1}} \cdots h_{i_{k_i}}^{s_{i_{k_i}}} = e_h,$$
   where e_H is the identity element in H, then there is a unique homomorphism $\phi : G \to H$ such that $\phi(x_i) = h_i$.

9. Show that $\mathbb{Z} \oplus \mathbb{Z}$ has presentation $\langle x, y | xyx^{-1}y^{-1}\rangle$.

10. Show that the dihedral group D_n has presentation $\langle x, y | x^n, y^2, xyxy\rangle$.

11. Consider the rational numbers $\mathbb{Q}$ as a group under addition. Show that $\mathbb{Q}$ has presentation
    $$\langle x_i, i = 1, 2, 3, \ldots | x_n^n = x_{n-1}, i > 1\rangle.$$
    Hint: try to construct a map by sending x_i to $\frac{1}{n!}$.

12. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then show $(g \circ f)_* = g_* \circ f_*$, where f_* denotes the homomorphism induced on the fundamental group by f.

13. If A is a subspace of X we say X *deformation retracts to* A if there is a continuous function $F : X \times [0,1] \to X$ so that $F_0 = id_X$, the image of F_1 is contained in A, and $F_1|_A = id_A$ for all t. Here $F_t : X \to X : x \mapsto F(x,t)$. (So the F_t "deform" X to A without moving A.) Show that if X deformation retracts onto A then X and A are homotopy equivalent.