III Manifolds

A. Definitions and first examples

a topological space M is called an n-manifold if it is

1) Hausdorff
2) 2^{nd} countable
3) each point $p \in M$ has an open neighborhood U homeomorphic to an open set V in $\mathbb{R}^n$

recall from homework 2 this means M has a countable basis

Remarks:

1) it can be shown that any n-manifold can be embedded in $\mathbb{R}^N$ for some N. (Some people include this in the definition of manifold, in which case 1) and 2) can be omitted since they are automatic)
2) n-manifolds are metric spaces
3) the idea is than an n-manifold is "locally Euclidean" (conditions 1) and 2) are just to avoid pathological examples)
4) 2-manifolds are also called surfaces
5) the homeomorphism $\phi : U \to V$ from 3) is called a coordinate chart

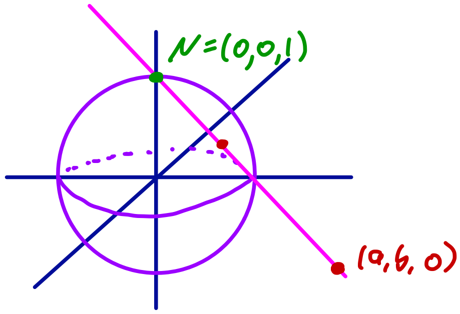
examples:

1) $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ is a surface

(Why is it Hausdorff and 2^{nd} countable?)

earlier we discussed coordinate charts of the form $(x,y) \mapsto (x, y, \sqrt{1-x^2-y^2})$

here we give a different approach



given (a,b) in xy-plane

let $\ell_{(a,b)}$ = line through $(a,b,0)$ and $N=(0,0,1)$

so $\ell_{(a,b)}$ is parameterized by

$$s(a,b,0)+(1-s)(0,0,1)$$
$$\shortparallel$$
$$(sa,\, sb,\, 1-s)$$

$\ell_{(a,b)} \cap S^2$: $\quad (as)^2+(bs)^2+(1-s)^2=1$

$$(a^2+b^2+1)s^2-2s=0$$

so intersection happens when $s=0$ or $s=\dfrac{2}{1+a^2+b^2}$

so $\ell_{(a,b)} \cap S^2$ at a unique point other than N

i.e at $\left(\dfrac{2a}{1+a^2+b^2},\ \dfrac{2b}{1+a^2+b^2},\ 1-\dfrac{2}{1+a^2+b^2}\right)$

so set $V=\mathbb{R}^2$ and $U=S^2-\{N\}$

then $\pi_N : V \to U : (a,b) \longmapsto \dfrac{2}{1+a^2+b^2}\left(2a,\, 2b,\, a^2+b^2-1\right)$

is a continuous map

to see π_N a homeomorphism we can construct π_N^{-1}

<u>exercise</u>: show $\pi_N^{-1}(x,y,z)=\left(\dfrac{x}{1-z},\ \dfrac{y}{1-z}\right)$

π_N^{-1} is called <u>stereographic projection</u> and

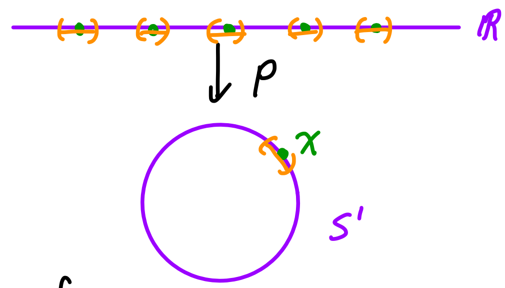
π_N is called <u>stereographic coordinates</u>

<u>note</u>: this shows S^2 is just $\mathbb{R}^2$ with one point added ("at infinity")

using $S=(0,0,-1)$ you can get a coordinate chart about N

exercise: Show S^n is an n-manifold by writting down stereographic coordinates

2) S^1 is a 1-manifold



consider $p : \mathbb{R} \to S^1$

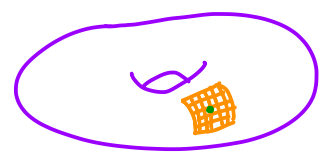
$x \longmapsto (\cos 2\pi x, \sin 2\pi x)$

given any $x \in S^1$ there is a small nbhd U of x s.t. $p^{-1}(U)$ = union of intervals

p restricted to each of these intervals is a homeomorphism

so S^1 is a 1-manifold

3) $T^2 = S^1 \times S^1$ is a surface



$(x_0, y_0) \in S^1 \times S^1$

x_0 has a nbhd $I \cong (a,b)$ in S^1

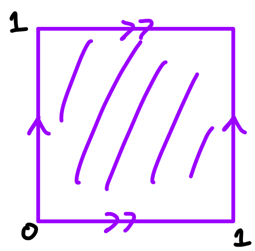
y_0 has a nbhd $J \cong (c,d)$ in S^1

so (x_0, y_0) has a nbhd $I \times J$ homeo to $(a,b) \times (c,d) \subset \mathbb{R}^2$

exercise: Show that the product of an n-manifold and an m-manifold is an $(n+m)$-manifold

4) T^2 again

recall from Section II.F, T^2 is a quotient space of

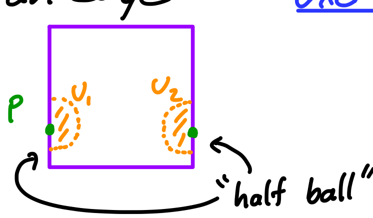


where opposite sides are identified

clearly any point $p \in (0,1) \times (0,1)$ has a nbhd homeomorphic to an open set in $\mathbb{R}^2$

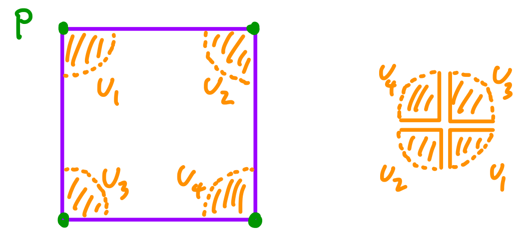
now if p is on an edge

exercise: $U_1 \cup U_2 / \sim \;\cong$ open ball



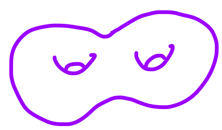
(similarly for other edge)

if p is a corner point

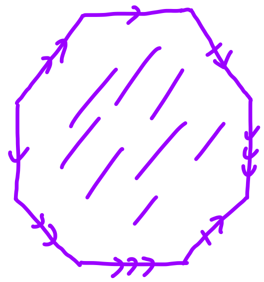


so T^2 is a surface (Why is it Hausdorff and 2^{nd} countable?)

5) In Section II.F we saw



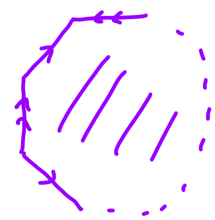
homeo to



exercise: as in example 4) check this is a surface

similarly

homeo to

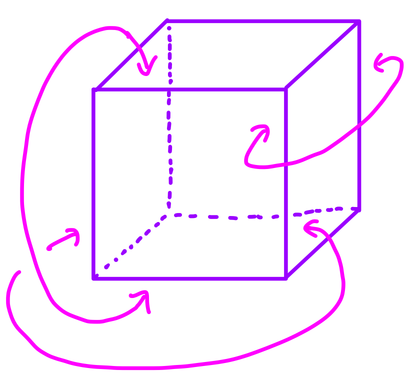


is a surface too

quotient of 4g-gon

6) $T^3 = S^1 \times S^1 \times S^1$ is a 3-manifold (by exercise above)

but also can consider $C = \text{cube} = [0,1]\times[0,1]\times[0,1]$



identify opposite sides by translation

exercise: 1) $C/\sim$ is T^3

2) show $C/\sim$ is a 3-manifold like T^2 in example 4)

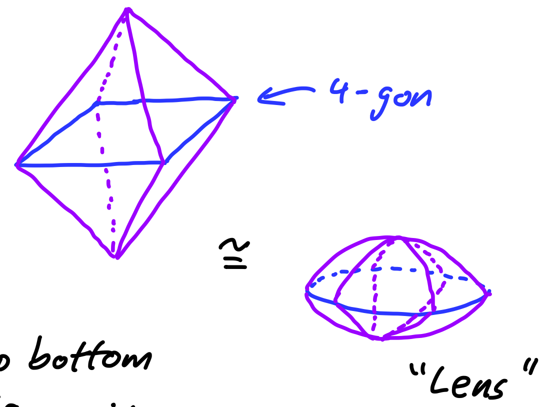
7) lens spaces $L(p,q)$

$p > q > 0$ rel. prime

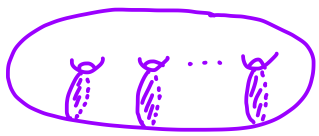
let $P =$ "suspension of p-gon"

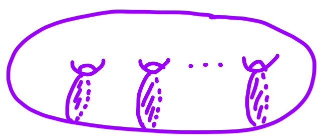
$L(p,q) = P/\sim$

where $\sim$ glues top to bottom after a $\frac{2\pi q}{p}$ twist



exercise: Show these are 3-manifolds

8) let $V_1 =$  3-dim'l stuff inside Σ_g

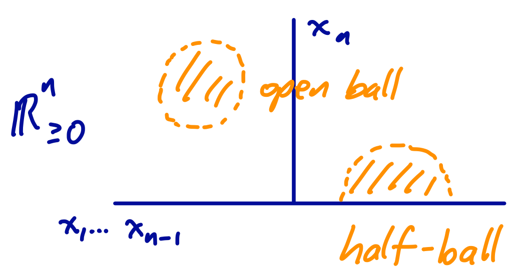
$V_2 =$ 

let $f : \partial V_1 \to \partial V_2$ be a homeomorphism, where $\partial V_1 = \Sigma_g$ and $\partial V_2 = \Sigma_g$

exercise: $M = V_1 \cup_f V_2$ is a 3-manifold

(all oriented, compact 3-manifolds are obtained this way!)

A second countable, Hausdorff space M is an n-manifold with boundary if each point $p \in M$ has a nbhd U_p homeomorphic to an open set in $\mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$



the boundary of M is

$$\partial M = \{ p \in M \mid p \text{ only has nbhd homeo. to nbhd of } (x_1, \dots, x_{n-1}, 0) \text{ in } \mathbb{R}^n_{\geq 0} \text{ and } p \text{ maps to point with } x_n = 0 \}$$

the interior of M is

$$\operatorname{int} M = M - \partial M$$

Important Facts:

1) no open set in $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{R}^m$ if $m \neq n$
2) no open nbhd of $(x_1, \dots x_{n-1}, 0)$ in $\mathbb{R}^n_{\geq 0}$ is homeomorphic to an open set in $\mathbb{R}^n$

Remarks:

1) $\Rightarrow$ if M is an n-manifold it is not an m-manifold for any $n \neq m$ (if $M \neq \emptyset$)
2) $\Rightarrow$ $\operatorname{int} M = \{ p \in M \mid p \text{ has a nbhd homeo. to an open set in } \mathbb{R}^n \}$

exercise:

if M is an n-manifold with boundary, then

1) ∂M is an $(n-1)$-manifold
2) $\operatorname{int} M$ is an n-manifold
3) $\partial(\partial M) = \emptyset$, $\partial(\operatorname{int} M) = \emptyset$, $\operatorname{int}(\partial M) = \partial M$, and $\operatorname{int}(\operatorname{int} M) = \operatorname{int} M$

B. 1-manifolds:

Th$^{\text{m}}$ 1:

If M is a connected 1-manifold, then M is homeomorphic to

1) S^1 if M is compact and without boundary
2) $[0,1]$ if M is compact and $\partial M \neq \emptyset$

3) $[0,1)$ if M is non-compact and $\partial M \neq \emptyset$, or

4) $(0,1) \cong \mathbb{R}$ if M non-compact without boundary

so we completely understand 1-manifolds!

the proof is not hard and can be found in many books/courses in topology (we skip the proof)

now what are symmetries of compact 1-manifolds (that is what are homeomorphisms)

two homeomorphisms

$$f_0, f_1 : X \to X$$

of a topological space are called isotopic if there is a homeomorphism

$$F : X \times [0,1] \to X \times [0,1]$$

with $F(x,t) = (F_t(x), t)$ and $F_0 = f_0$, $F_1 = f_1$

this implies $F_t : X \to X$ is a homeomorphism

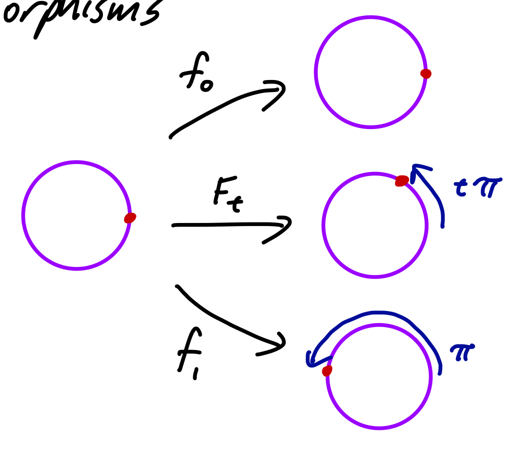
so two homeomorphisms are isotopic if you can continuously deform one into the other through homeomorphisms

example: $f_0 : S^1 \to S^1$ identity

$f_1 : S^1 \to S^1$ rotation by π

let F_t = rotation by πt

so $F_t(x,t) = (F_t(x), t)$ is an isotopy from f_0 to f_1



lemma 2:

1) any homeomorphism $f : [0,1] \to [0,1]$ is isotopic to

$$id : [0,1] \to [0,1] : x \mapsto x \quad \text{or}$$

$$r : [0,1] \to [0,1] : x \mapsto 1-x$$

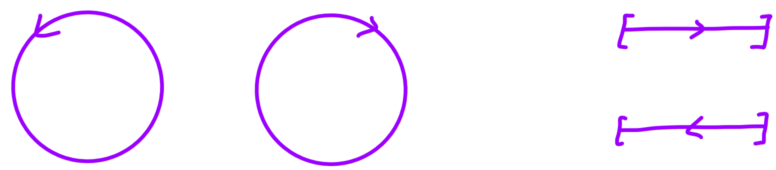
2) any homeomorphism $f : S^1 \to S^1$ is isotopic to

$$id : S^1 \to S^1 : (x,y) \mapsto (x,y)$$

$$r : S^1 \to S^1 : (x,y) \mapsto (x,-y)$$

so we completely understand homeomorphisms of compact 1-manifolds upto isotopy

an orientation on a 1-manifold is a choice of direction



note: lemma 2 says homeomorphisms of $[0,1]$ or S^1 are isotopic iff they both preserve or reverse orientations

Sketch of Proof:

let $f:[0,1] \to [0,1]$ be an orientation preserving homeomorphism

note: $f(0)=0, \ f(1)=1$

set $F_t(x) = (1-t)f(x) + tx$

check this gives an isotopy

for $f: S^1 \to S^1$ orientation preserving

use rotation of S^1 to isotope f until $f((1,0)) = (1,0)$

recall we have a quotient map $q: [0,1] \to S^1$

from this we get $\tilde{f}: [0,1] \to [0,1]$ an orientation preserving homeo.

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\tilde{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\circ"] & {[0,1]} \arrow[d, "q"] \\
S^1 \arrow[r, "f"] & S^1
\end{tikzcd}$$

so we have an isotopy $\tilde{F}_t: [0,1] \to [0,1]$ from $\tilde{f}$ to id

let $\bar{F}_t = q \circ F_t : [0,1] \to S^1$

since $\bar{F}_t$ sends 0 and 1 to same point it induces a map

$$F_t : S^1 \to S^1$$

quotient space theory says F_t is continuous and it is clearly a bijection

$\therefore F_t$ is a homeomorphism since S^1 is compact and Hausdorff

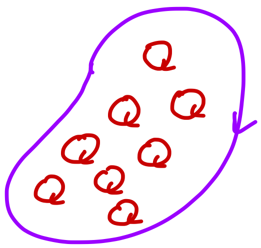
(Th^{m} II. 18)

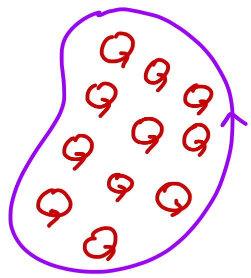
exercise: think about the orientation reversing case

C. 2-manifolds

Can think of an orientation on a domain in $\mathbb{R}^2$ as a (consistent) choice of orientation on a small closed curve at each point

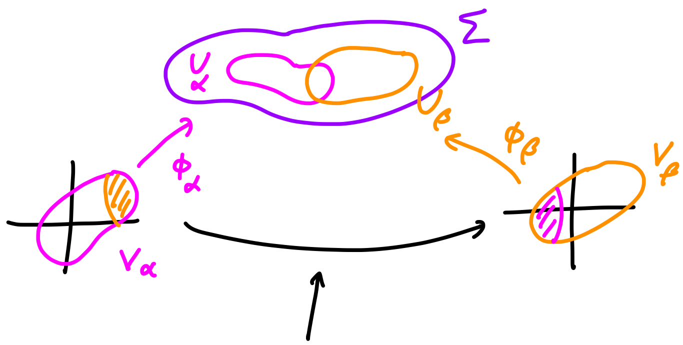
image of S^1


clockwise


counter clockwise

note: this induces an orientation on the boundary

a surface is oriented if given any coordinate charts $\{\phi_\alpha : V_\alpha \to U_\alpha\}_{\alpha\in A}$ such that $\Sigma = \bigcup_{\alpha\in A} U_\alpha$ there is a choice of orientations on the V_α such that whenever $U_\alpha \cap U_\beta \neq \emptyset$ we have



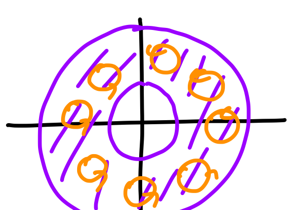
the map $\phi_\beta^{-1}\circ\phi_\alpha : \phi_\alpha^{-1}(U_\alpha\cap U_\beta) \to \phi_\beta^{-1}(U_\alpha\cap U_\beta)$ sends the orientation on V_α to the one on V_β (note $\phi_\beta^{-1}\circ\phi_\alpha$ sends closed curves to closed curves)

if Σ cannot be oriented it is called non-orientable

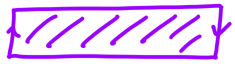
examples:

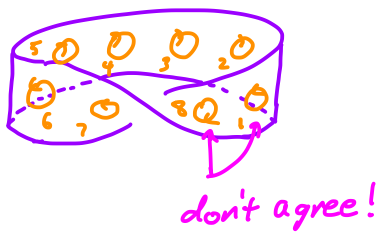
1) the annulus $A = S^1\times[0,1]$ can be oriented

eg



now any coordinate charts $\phi: U\to V$ for A we use to orient V

2) the Möbius band $M =$ 



exercise: 1) find 2 charts on M so that there is no way to satisfy the orientation condition above (ie. rigorously show M is not orientable)

2) Show a surface is not orientable
$$\Leftrightarrow$$
it contains a Möbius band

Given two surfaces Σ_1 and Σ_2

let D_i be a disk in Σ_i

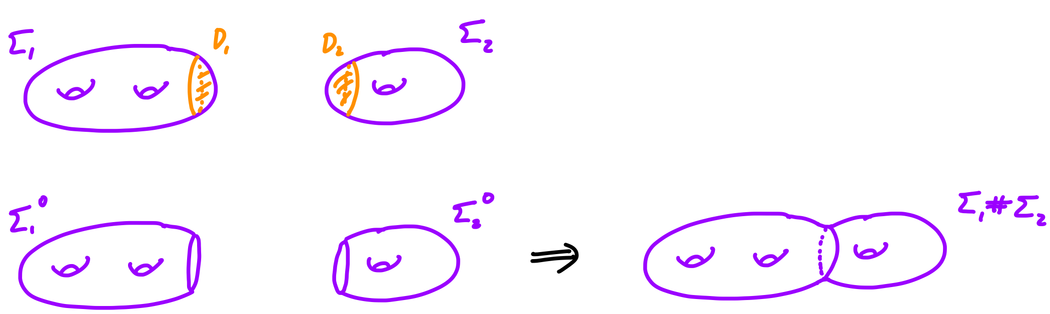
(if Σ_i is oriented give D_i orientation induced from Σ_i
otherwise choose an arbitrary orientation on D_i)

let $\Sigma_i^\circ = \Sigma_i - (\mathrm{int}\, D_i)$

let $f : \partial D_1 \to \partial D_2$ be an orientation reversing homeomorphism
$$\begin{array}{ccc} \partial D_1 & & \partial D_2 \\ \cap & & \cap \\ \partial \Sigma_1^\circ & & \partial \Sigma_2^\circ \end{array}$$

the connected sum of Σ_1 and Σ_2 is
$$\Sigma_1 \# \Sigma_2 = \Sigma_1^\circ \cup_f \Sigma_2^\circ$$

example:



exercise: if Σ_1 and Σ_2 are oriented then so is $\Sigma_1 \# \Sigma_2$

Th^m 3:

> the connected sum of two connected surfaces is well-defined

to see this we need to see that the construction is independent of

1) disks D_1 and D_2 used, and
2) homeomorphism f

for these we have

lemma 4:

> let D and D' be two disks in Σ (if Σ oriented then, orient D and D' with this orientation)
>
> Then there is a homeomorphism
>
> $$\phi : (\Sigma - \mathrm{int}\, D) \longrightarrow (\Sigma - \mathrm{int}\, D')$$
>
> that preserves the orientation of the boundary

lemma 5:

> let M and N be two manifolds with boundary and
>
> $$f_0, f_1 : \partial M \to \partial N$$
>
> two homeomorphisms.
>
> if f_0 is isotopic to f_1, then $M \cup_{f_0} N \cong M \cup_{f_1} N$

very important lemma!

Remark: lemma 4 is similar to Exercise 9 on Homework 3

so should be believable

for the sake of time we skip the proof

Proof of Th^m 3:

let $D_1, D_1' \subset \Sigma_1$ and $D_2, D_2' \subset \Sigma_2$ be disks

and $f : \partial D_1 \longrightarrow \partial D_2$, $f' : \partial D_1' \longrightarrow \partial D_2'$ be orientation reversing homeo.s

from lemma 4 we get homeomorphisms

$$\phi : \underbrace{(\Sigma_1 - \mathrm{int}\, D_1)}_{\Sigma_1^{\circ}} \longrightarrow \underbrace{(\Sigma_1 - \mathrm{int}\, D_1')}_{\Sigma_1^{\circ\circ}}$$

and

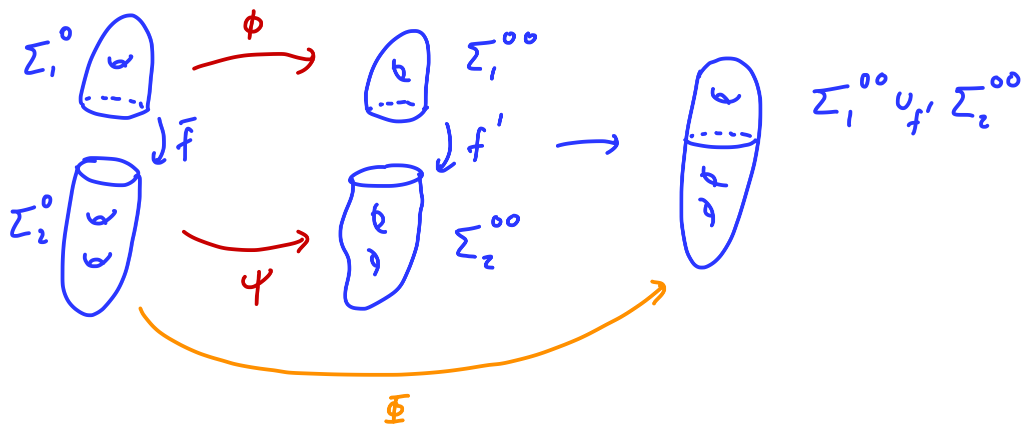
$$\psi : \underbrace{(\Sigma_2 - \mathrm{int}\, D_2)}_{\Sigma_2^{o}} \to \underbrace{(\Sigma_2 - \mathrm{int}\, D_2')}_{\Sigma_2^{oo}}$$

let $\bar f = \psi^{-1} \circ f' \circ \phi : \underset{\partial \Sigma_1^{o}}{\underset{\cap}{\partial D_1}} \to \underset{\partial \Sigma_2^{o}}{\underset{\cap}{\partial D_2}}$ note: $\bar f$ is an orientation reversing homeomorphism

so f and $\bar f$ are isotopic by lemma 2

thus $\Sigma_1^{o} \cup_f \Sigma_2^{o} \cong \Sigma_1^{o} \cup_{\bar f} \Sigma_2^{o}$ by lemma 5

but



Φ induces a homeomorphism

$$\Sigma_1^{o} \cup_{\bar f} \Sigma_2^{o} \to \Sigma_1^{oo} \cup_{f'} \Sigma_2^{oo}$$

on the quotient space (check this!)

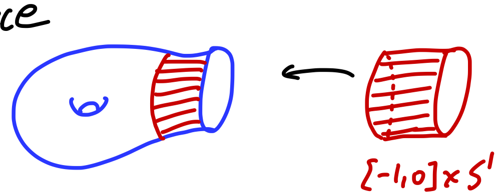
to prove lemma 5 we need

lemma 6: If M is a manifold with boundary, then there is an embedding

$$\phi : ([-1,0] \times \partial M) \to M$$

such that $\phi(\{0\} \times \partial M) = \partial M$

for a surface

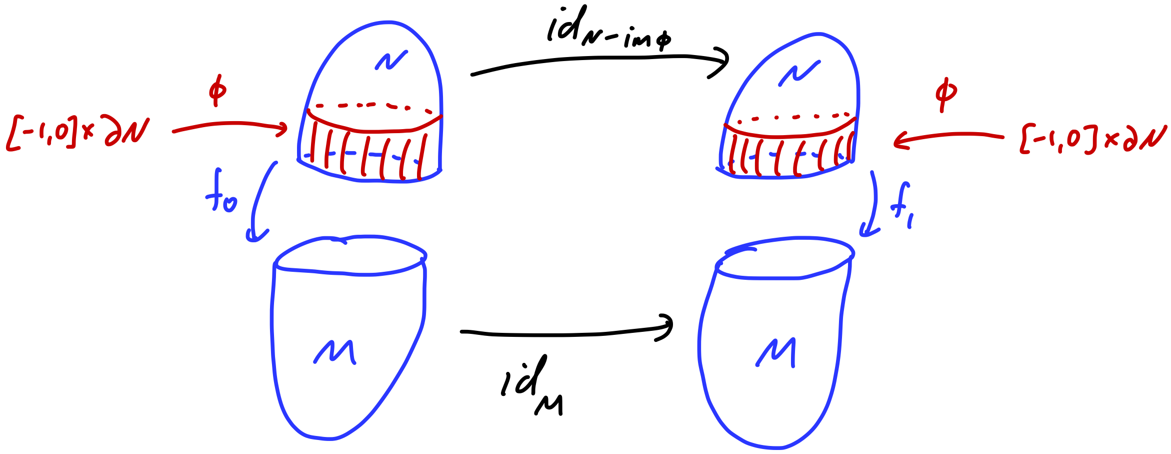


this is called a collar neighborhood of boundary for surfaces this is intuitively obvious

this lemma is easy to prove using ideas from graduate math courses, but we will not prove it here

Proof of lemma 5:

we need to build a homeomorphism



we want to extend over $\mathrm{im}(\phi)$ to get a homeomorphism on the quotient space

for this let $F: ([0,1]\times\partial M) \to ([0,1]\times\partial N)$

$$(t,p) \longmapsto (t, F_t(p))$$

be the isotopy from f_0 to f_1

note: $G: ([0,1]\times\partial N) \to ([0,1]\times\partial N)$

$$(t,p) \longmapsto (t, f_1^{-1}\circ F_t(p))$$ call this G_t

is an isotopy from $f_1^{-1}\circ f_0$ to $id_{\partial N}$

set $\tilde{G}: ([-1,0]\times\partial N) \to ([-1,0]\times\partial N)$

$$(t,p) \longmapsto (t, G_{-t}(p))$$

then we can extend the map above by

$$\mathrm{im}\,\phi \longrightarrow \mathrm{im}\,\phi$$
$$p \longmapsto \phi\circ\tilde{G}\circ\phi^{-1}(p)$$

you can easily check this gives a homeomorphism

$$M\cup_{f_0} N \text{ to } M\cup_{f_1} N$$

let's build some surfaces

if M is the Möbius band and D^2 is a disk, then $\partial M = S^1$ and $\partial D^2 = S^1$

so choose a homeomorphism $\phi : \partial M \to \partial D^2$

just like in earlier examples

$$P = M \cup_f D^2$$

is a surface (without boundary)

it is called the projective plane

note: P is not orientable

exercise: 1) given  S^2 the unit sphere in $\mathbb{R}^3$

let $r : S^2 \to S^2 : (x,y,z) \mapsto (-x,-y,-z)$

say $p_1, p_2 \in S^2$ are equivalent if $r(p_1) = p_2$ $(\therefore r(p_2) = p_1)$

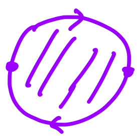
Show: $S^2/_\sim \cong P$

2)  D^2 unit disk in $\mathbb{R}^2$

let $r : \partial D^2 \to \partial D^2 : (x,y) \mapsto (-x,-y)$

define $\sim$ on ∂D^2 as above

Show $D^2/_\sim \cong P$

3) $P \cong$  identify edges so arrows match

now define: $\Sigma_0 = S^2$ 

$\Sigma_1 = T^2$ 

$\Sigma_2 = T^2 \# T^2$ 

$\vdots$

$\Sigma_n = \Sigma_{n-1} \# T^2$ 

$\vdots$

n holes

$\vdots$

and $N_1 = P$  called a "cross cap" we can't really draw in $\mathbb{R}^3$!

$N_2 = P \# P$ 

$\vdots$

$N_n = N_{n-1} \# P$ 

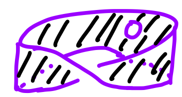
$\vdots$

now given n and m let $D_1 \dots D_m$ be m disjoint disks in Σ_n or N_n

then set
$$\Sigma_{n,m} = \Sigma_n - \bigcup_{i=1}^{m} \operatorname{int} D_i$$

$\Sigma_{1,2}$

$$N_{n,m} = N_n - \bigcup_{i=1}^{m} D_i$$



$N_{1,2}$

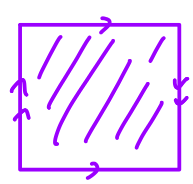
Th$^{\underline{m}}$ 7:

> If Σ is any compact, connected surface (possibly with boundary) then there is some n and m such that Σ is homeomorphic to
>
> $\Sigma_{n,m}$ if Σ is orientable, or
>
> $N_{n,m}$ if Σ is not-orientable
>
> Moreover, $\Sigma_{n,m}$ and $\Sigma_{n',m'}$ (and $N_{n,m}$ and $N_{n',m'}$) are homeomorphic $\iff$ $n=n'$ and $m=m'$

Great theorem! a complete classification of surfaces!

but we would like to do better since right now it is not clear what surface on the list is

$$\Sigma_{n,m} \# N_{n',m'} \;?\; \text{ or}$$
$$\Sigma_{n,m} \# \Sigma_{n',m'} \;?\; \text{ or}$$

$K =$  ?

Klein bottle

Remarks:

1) You can find a "standard" proof of this in most topology books/courses so we do not give that proof here but discuss a non-standard "surgery" proof

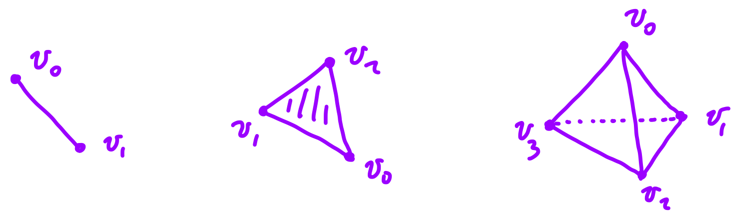
2) Classification of non-compact surfaces is also known, but very complicated and we will not need it

To improve (and prove) Th$^{\underline{m}}$ 7 we need the Euler characteristic given $k+1$ points, $v_0, \ldots v_k$, in $\mathbb{R}^N$ (some large N) in general position (that is no 3 points lie on a line, no 4 on a plane, ...)

then a k-simplex is the set

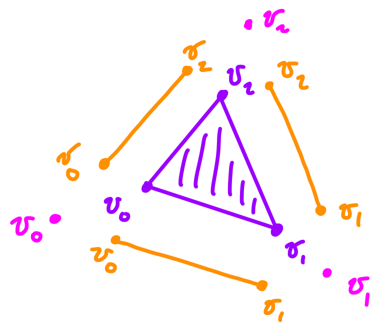
$$\Delta_k = \{ \lambda_0 v_0 + \ldots + \lambda_k v_k \mid \lambda_i \geq 0 \text{ and } \lambda_0 + \ldots + \lambda_k = 1 \}$$

examples:



a face of a simplex is a subsimplex formed by discarding some verticies

example:

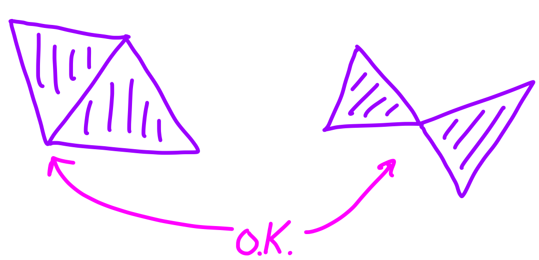


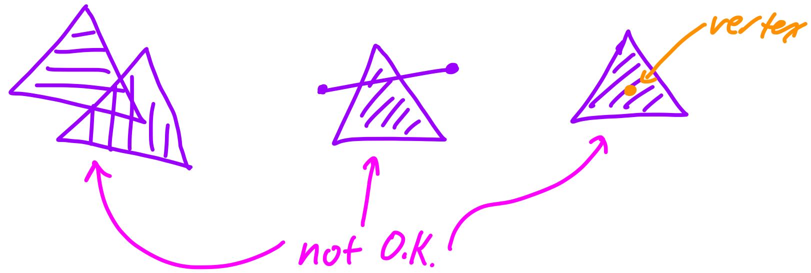
a simplicial complex is a finite collection of simplicies in some $\mathbb{R}^N$ such that

a) if a simplex is in the collection then so are all of its faces

b) if two simplicies intersect then they do so in one common face (and its subfaces)

example:

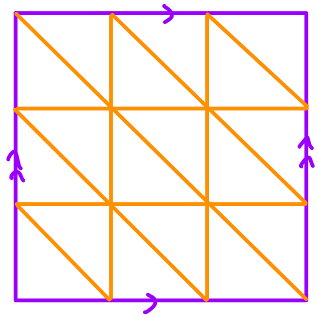


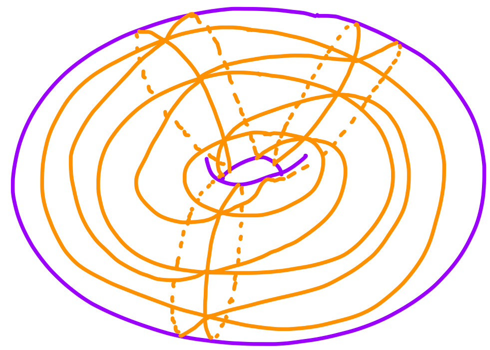


a triangulation of a topological space X is a simplicial complex K together with a homeomorphism $h: K \to X$

example:

T^2





Hard Theorem (Radó 1925):

$$\boxed{\text{any surface has a triangulation}}$$

let K be a simplicial complex (with no n-simplicies for $n \geq k$)

the Euler characteristic of K is

cell = simplex

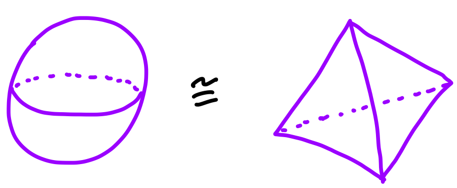
$$\chi(K) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) + \ldots + (-1)^k \#(k\text{-cells})$$
$$= \sum_{i=0}^{k} (-1)^i \#(i\text{-cells})$$

if X is a topological space homeomorphic to K then the Euler characteristic of X is

$$\chi(X) = \chi(K)$$

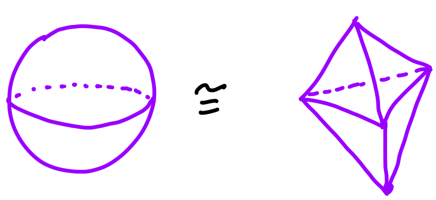
example:

1) S^2

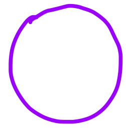
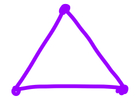


$$\chi(S^2) = 4 - 6 + 4 = 2$$

2) S^2



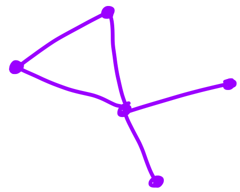
$$\chi(S^2) = 5 - 9 + 6 = 2$$

3) S^1  $\cong$ 

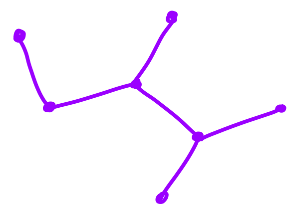
$$\chi(S^1) = 3 - 3 = 0$$

4) $[0,1]$  $\chi([0,1]) = 2 - 1 = 1$

exercise: a graph is a simplicial complex with only 0 and 1-simplicies



graph



tree ← no loops in graph (and connected)

1) Show if T is a tree, then $\chi(T) = 1$

Hint: induct on the number of 0-simplicies

2) if G is a connected graph, then show

$$\chi(G) \leq 1$$

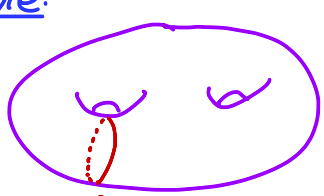
with equality $\Leftrightarrow$ G is a tree

Remark: It is not clear the Euler characteristic is well-defined for a topological space, but it is!
We will not prove this here, but it is easy once you define homology.

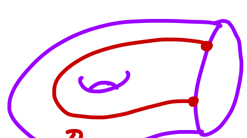
an embedding $e: M \to N$ of a compact manifolds is proper if

$$e(\partial M) \subset \partial N \quad \text{and}$$
$$e(\operatorname{int} M) \subset \operatorname{int} N$$

example:

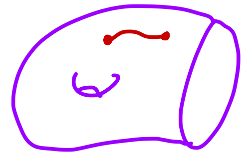


proper



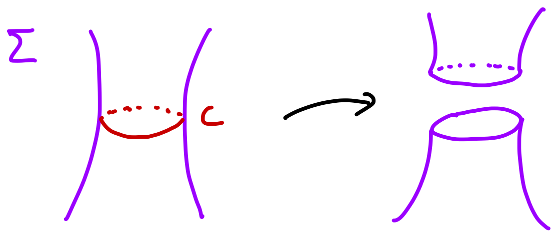
proper

not proper



given a proper empedded 1-manifold C in a surface Σ we can

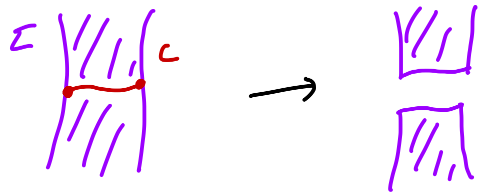
cut Σ along C



i.e. consider $\Sigma - C$ then put back two copies of C

denote this by

$\Sigma \setminus C$

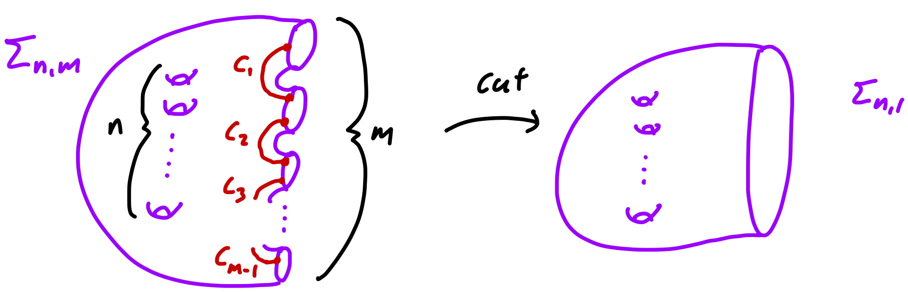


lemma 8:

> If C is a properly embedded 1-manifold in the surface Σ, then
> $$\chi(\Sigma \setminus C) = \chi(\Sigma) + \chi(C)$$

Proof: note the verticies and edges in C are counted once in Σ and twice in $\Sigma \setminus C$

let's compute $\chi(\Sigma_{n,m})$ for $m \geq 1$



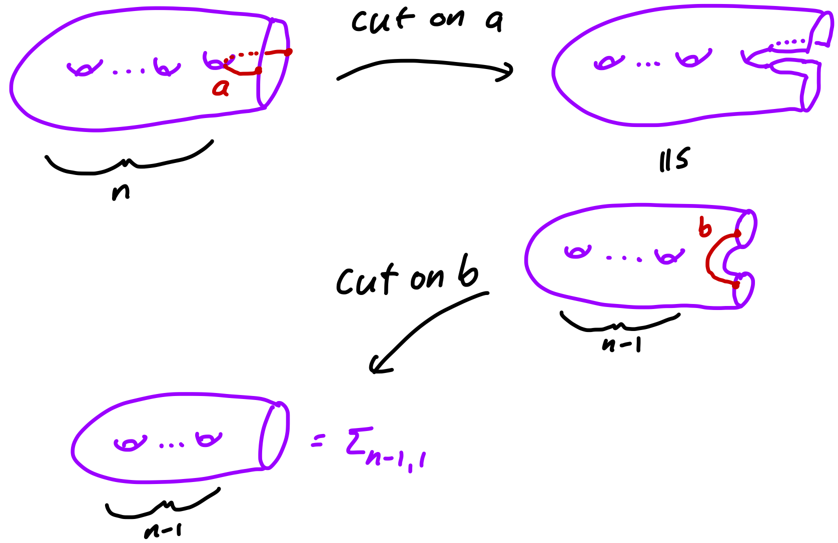
note: $(m-1)$ arcs $c_1, \dots, c_{m-1}$ to cut and get $\Sigma_{n,1}$

$\chi(\text{arc}) = 1$

so $\chi(\Sigma_{m,n}) = \chi(\Sigma_{n,1}) - (m-1)$

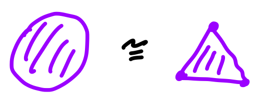
now

$\Sigma_{n,1}$



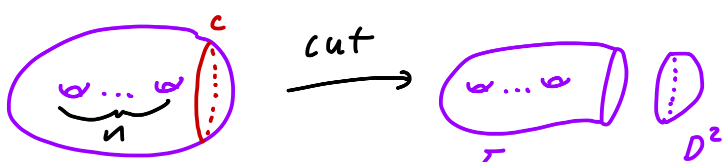
so $\chi(\Sigma_{n,1}) = \chi(\Sigma_{n-1,1}) - 2$

$$= \cdots = \chi(\Sigma_{0,1}) - 2n$$

and $\Sigma_{0,1} =$  $\quad \chi(\Sigma_{0,1}) = 3-3+1 = 1$

so $\chi(\Sigma_{n,m}) = -m+1-2n+1 = 2-2n-m$ for $m \geq 1$

now



$$\begin{aligned} \therefore \chi(\Sigma_n) &= \chi(\Sigma_n \setminus c) - \chi(c) \\ &= \chi(\Sigma_{n,1}) + \chi(D^2) - \chi(c) \\ &= 1-2n+1-0 \\ &= 2-2n \end{aligned}$$

so we have

$$\boxed{\begin{array}{ll} \chi(\Sigma_{n,m}) = 2-2n-m & \text{for all } n,m \\ \chi(N_{n,m}) = 2-n-m & \text{"} \quad \text{"} \end{array}}$$

$\leftarrow$ check this

Remark: easy way to compute χ

$$\boxed{\chi(\Sigma) = \begin{cases} 1 - \#\text{arcs to cut } \Sigma \text{ to a disk} & \text{if } \partial\Sigma \neq \emptyset \\ 2 - \#\text{arcs to cut } (\Sigma - D^2) \text{ to a disk} & \text{if } \partial\Sigma = \emptyset \end{cases}}$$

exercise: $\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$

for a topological space X, let $|X|$ denote the number of connected components

Thm 9:

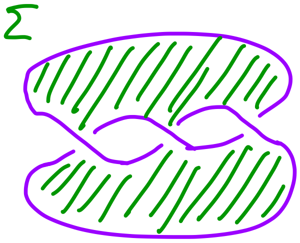
two compact, connected surfaces Σ_1 and Σ_2 are homeomorphic

$\Leftrightarrow$

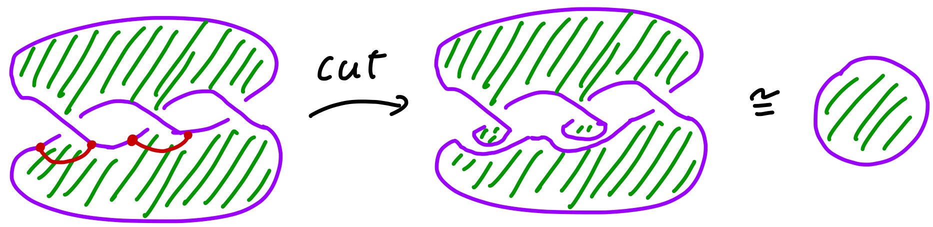
$\chi(\Sigma_1) = \chi(\Sigma_2)$, $|\partial \Sigma_1| = |\partial \Sigma_2|$ and Σ_1 and Σ_2 are both orientable or both are non-orientable

Moreover, any compact, connected surface is homeomorphic to $\Sigma_{n,m}$ or $N_{n,m}$

example: What surface is Σ



note: cut on 2 arcs to get a disk

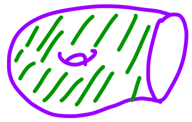


so $\chi(\Sigma) = 1 - 2 = -1$

$|\partial \Sigma| = 1$

the surface is orientable since as you go around any loop on Σ you don't have an odd number of half twists (similarly, you could note that the surface has "two sides" that is you could make it out of paper and color the sides with two colors) so $\Sigma \cong \Sigma_{n,1}$ for some n

$$-1 = \chi(\Sigma_{n,1}) = 2-2n-1 = 1-2n \Rightarrow n=1$$

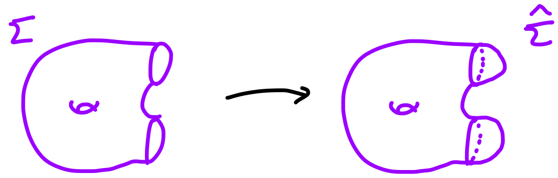
so $\Sigma \cong \Sigma_{1,1}$  it is just embedded in $\mathbb{R}^3$ strangely!

Sketch of proof of Th^m 9 (and hence Th^m 7):

we first reduce to the closed case with

exercise: let Σ and Σ' be surfaces with $|\partial\Sigma| = |\partial\Sigma'|$

let $\hat{\Sigma}$ and $\hat{\Sigma}'$ be Σ and Σ' with disks glued to each boundary component



(e.g. $\hat{\Sigma} = \Sigma \cup_{\phi_i} (D_1 \cup \dots \cup D_{|\partial\Sigma|})$

where $\phi_i : \partial D_i \to C_i$ is a homeo

and $\partial\Sigma = C_1 \cup \dots \cup C_{|\partial\Sigma|}$)

Then show Σ homeo to Σ' $\Leftrightarrow$ $\hat{\Sigma}$ homeo to $\hat{\Sigma}'$

Hint: ($\Rightarrow$) uses lemmas 2 and 5

($\Leftarrow$) is a generalization of lemma 4

thus from exercise we see Th^m 9 is true if it is true for compact, connected surfaces without boundary

note all the Σ_n and N_n are different (since they have different Euler characteristics or one is orientable and other not)

so all we have to do is show a compact, connected surface Σ without boundary is homeomorphic to Σ_n or N_n for some n

Claim 1: $\boxed{\chi(\Sigma) \leq 2 \text{ and } \chi(\Sigma) = 2 \Leftrightarrow \Sigma \cong S^2}$

Claim 2: if $\Sigma \not\cong S^2$, then there is an embeding $\phi: S^1 \to \Sigma$ such that $\Sigma \setminus \phi(S^1)$ is connected

moreover, a) Σ orientable $\Rightarrow$ $\phi(S^1)$ has a neighborhood homeo to $[-1,1]\times S^1$ with $\{0\}\times S^1 = \phi(S^1)$

b) Σ non-orientable $\Rightarrow$ we may assume $\phi(S^1)$ has a nbhd homeo to a Möbius band and $\Sigma - \phi(S^1)$ is either D^2 or is non-orientable we see the th$^{\underline{m}}$ follows from these claims

"Induct" on $\chi(\Sigma)$

note Claim 1 says th$^{\underline{m}}$ true for $\chi(\Sigma)=2$

we inductively assume th$^{\underline{m}}$ for all surfaces with $\chi(\Sigma) \geq k+1$

and then prove for $\chi(\Sigma)=k$

(kind of a "reverse induction" could be "normal induction"

by inducting on $2-\chi(\Sigma)$)

Assume Σ non-orientable

then by Claim 2, $\exists$ a Möbius band M in Σ (M is nbhd of $\phi(S^1)$)

let $\Sigma' = \overline{\Sigma - M} \cup_f D^2$ where $f: \partial D^2 \to \partial(\overline{\Sigma - M})$ is a homeo.

<u>note</u>: Σ' is well defined by lemmas 2 and 5

we say Σ' is obtained from Σ by <u>surgery on $\phi(S^1)$</u> i.e. remove something (M) and glue back something (D^2)

<u>note</u>: 1) Σ' is non-orientable or S^2

by Claim 2(b)

2) recall $P = M = D^2$ projective plane

so $\Sigma = \Sigma' \# P$

3) $\chi(\overline{\Sigma - M}) = \chi(\overline{\Sigma - M}) + \overset{=0}{\chi(M)}$

$= \chi(\Sigma \setminus \partial M) = \chi(\Sigma) - \overset{=0}{\chi(S^1)} = \chi(\Sigma)$

$\chi(\Sigma') = \chi(\Sigma' \setminus \partial D^2) - \overset{=0}{\chi(\partial D^2)}$

$= \chi(\overline{\Sigma - M}) + \chi(D^2) = \chi(\Sigma) + 1$

so by induction on χ, Σ' is N_n, some n, or S^2

$$\therefore \Sigma = \Sigma' \# P = \begin{cases} N_n \# P = N_{n+1} \\ S^2 \# P = P = N_1 \end{cases} \checkmark$$

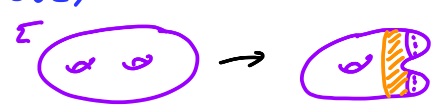
Assume Σ orientable

so by Claim 2, $\exists$ a annulus $A \subset \Sigma$ (A is a nbhd of $\phi(S^1)$)

let $\Sigma' = (\overline{\Sigma - A}) \cup_f (D^2 \cup D^2)$ where $f: (\partial D^2 \cup \partial D^2) \to \partial(\overline{\Sigma - A})$ is a homeo.

Σ' is said to be obtained from Σ by <u>surgery on $\phi(S^1)$</u>

<u>note</u>: 1) Σ' is orientable (exercise)

2) $\Sigma = \Sigma' \# T^2$ (exercise )

3) $\chi(\Sigma') = \chi(\Sigma) + 2$ (exercise)

so as above $\Sigma' \cong \Sigma_n$ for some n

$$\therefore \Sigma \cong \Sigma_n \# T^2 = \Sigma_{n+1}$$

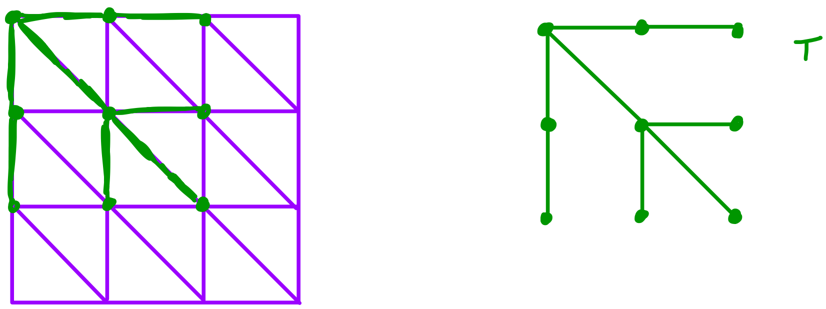
Idea for Claim 1:

let $\mathcal{T}$ be a triangulation of Σ

choose a maximal tree T in 1-skeleton

(i.e. contains all verticies and if you add another edge then no longer a tree)

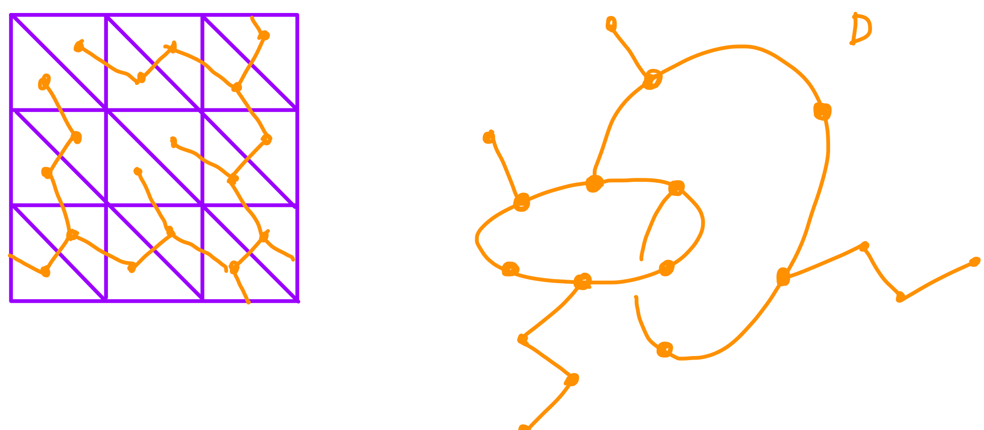
e.g.



let D be the dual graph, that is, D has

1) a vertex for in the center of each 2-simplex
2) two verticies are connected by an edge $\Leftrightarrow$ the 2-simplicies share an edge not in T

e.g.



exercise: D is connected

let v, e, f be the number of verticies, edges, and faces of $\mathcal{T}$ and v_T, e_T, v_D, e_D same for T and D

note: $v = v_T$ by construction

$e = e_T + e_D$ " "

$f = v_D$ " "

so $\chi(\Sigma) = v - e + f = v_T - e_T - e_D + v_D$
$= \chi(T) + \chi(D)$

from earlier exercise $\chi(\text{connected graph}) \leq 1$
with equality $\Leftrightarrow$ graph a tree

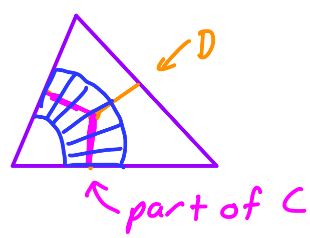
$\therefore \chi(\Sigma) = 1 + \chi(D) \leq 2$
with $= \Leftrightarrow D$ a tree

exercise: if D is a tree then show Σ is obtained by gluing 2 disks along their boundary by a homeo.
i.e. $\Sigma \cong S^2$
hint: neighborhoods of trees are disks

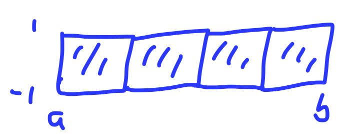
this proves Claim 1

Claim 2:
If D is not a tree then there is a loop in D.
Thus an embedding of $S^1 \hookrightarrow D \subset \Sigma$
let C be this loop
note: $C \cap (2\text{-simplex}) = \begin{cases} \emptyset \\ \text{interval } I \end{cases}$
so C has a neighborhood in each 2-simplex it hits of the form $I \times [-1,1]$

so a neighborhood of C is obtained by gluing many copies of $I \times [-1,1]$ along $(\partial I) \times [-1,1]$



exercise: This is homeomorphic to $[a,b] \times [-1,1]$ with $\{a\} \times [-1,1]$ glued to $\{b\} \times [-1,1]$ by a homeo



so neighborhood N is an annulus or Möbius band

if Σ orientable, must be an annulus. so done with Claim 2(a) now if Σ is non-orientable, then by definition there is an embedded Möbius band

so we can take N to be this Möbius band

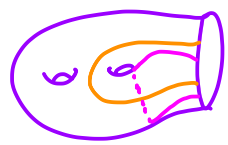
then if $\Sigma - N$ is non-orientable or D^2 we are done

if $\Sigma - N$ is orientable, then we know Σ' (= surgery on core of N) is Σ_n for some n (i.e. do classification of orientable surfaces first)

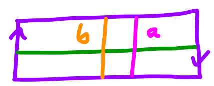
so $\Sigma = \Sigma_n \# P$

if $n=0$, then $\overline{\Sigma - N} = D^2$ so done

if $n>0$, then note



$\Sigma_n - D$



N = Möbius band

check a neighborhood of a and b are Möbius bands and so we can use one of these to prove Claim 2(b)

Remarks:

1) Use understanding of homeos $S^1 \to S^1$ to build surfaces (connect sums, surgery, ...)
2) Use embeddings of $S^1 \hookrightarrow$ surfaces to classify surfaces!