E. Free products and topology

a group G is called <u>indecomposable</u> if whenever $G \cong G_1 * G_2$ we must have G_1 or G_2 be the trivial group

example: 1) finite groups are indecomposable (since nontrivial free products are infinite)

both follow from homework

2) abelian groups are indecomposable (since nontrivial free products are non-abelian)

Can you always break a group into indecomposable pieces?

i.e. write $G \cong G_1 * \dots * G_n$ where G_i are indecomposable?

to study this we define the <u>rank</u> of G to be

$$\rho(G) = \min\{|X| : X \text{ a generating set for } G\}$$

cardinality of X

examples:

1) $\rho(\oplus_n \mathbb{Z}) = n$

to see this note we clearly have

$$\rho(\oplus_n \mathbb{Z}) \leq n$$

but if k elements generate $\oplus_n \mathbb{Z} \leq \mathbb{R}^n$

then k elements also span $\mathbb{R}^n$ as a vector space, so $k \leq n$

i.e. $\rho(\oplus_n \mathbb{Z}) = n$

2) If F_n is the free group generated by n elements then $\rho(F_n) = n$

indeed, clearly $\rho(F_n) \leq n$

note if $H \trianglelefteq G$, then $\rho(G/H) \leq \rho(G)$

since if $g_1, \dots, g_n$ generate G,

then $Hg_1, \dots, Hg_n$ generate G/H

now $\oplus_n \mathbb{Z} \cong F_n / [F_n, F_n]$ commutator subgroup

so $\rho(F_n) \geq \rho(\oplus_n \mathbb{Z}) \geq n$

$\therefore \rho(F_n) = n$

note: $F_n \cong F_{n-k} * F_k$ for all $0 \leq k \leq n$

so $\rho(F_n) = \rho(F_{n-k}) - \rho(F_k)$

we would like to generalize this to all groups!

for this we need

Theorem 7 (Grushko, 1940):

let F be a finitely generated free group and

$$\phi : F \to G_1 * G_2$$

be any surjective homomorphism

Then $F = F_1 * F_2$ where $\phi(F_i) = G_i$. $i=1,2$

before we prove this we notice some corollaries

Cor 8:

$$\rho(G_1 * G_2) = \rho(G_1) + \rho(G_2)$$

Proof: clearly $\rho(G_1 * G_2) \leq \rho(G_1) + \rho(G_2)$

(union of generating sets for G_1 and G_2 will generate $G_1 * G_2$)

suppose $\rho(G_1 * G_2) = n$

then there is a surjective homomorphism

$$\phi : F \to G_1 * G_2$$

where F is a free group of rank n

by Theorem 7, $F = F_1 * F_2$ where $\phi(F_i) = G_i$.

Fact: a subgroup of a free group is free

(we will prove this later)

so F_i is free and by the example above

we know $n = n_1 + n_2$ where $\rho(F_i) = n_i$

so $\rho(G_i) \leq n_i \quad i = 1, 2$

and $\rho(G_1 * G_2) = n = n_1 + n_2 \geq \rho(G_1) + \rho(G_2)$ ▦

Cor 9:

> If G is finitely generated, then
> $$G \cong \mathop{\ast}_{i=1}^{n} G_i$$
> where G_i is indecomposable

Proof: if G is indecomposable, then done $(n=1)$

if not $G = G_1 * G_2$ with $G_1, G_2 \neq \{1\}$

by Corollary 8 $\quad \rho(G_i) < \rho(G) \quad i = 1, 2$

$\therefore$ done by induction on $\rho(G)$

(since clearly $\rho(G) = 1 \Rightarrow G$ cyclic

$\therefore$ indecomposable) ▦

Remark: Cor 9 is <u>not</u> <u>true</u> for <u>non-finitely generated</u> groups

Question: Is there a group G that has a normal subgroup H such that $G/H \cong G$?

(can a quotient group be isomorphic to the original group?)

a group G is called **Hopfian** if:

$\phi: G \to G$ a surjective homomorphism
$\Rightarrow \phi$ an isomorphism

Remark: G Hopfian $\Leftrightarrow$ answer to question is No for G

examples:

1) any finite group is Hopfian

2) S^1 is not Hopfian — unit circle in $\mathbb{C}$

   e.g. let $\phi: S^1 \to S^1 : z \mapsto z^n$

   clearly ϕ is surjective

   but $\ker \phi = \{ e^{\frac{2\pi k i}{n}} \}_{k=0}^{n-1}$

3) $(\mathbb{R}, +)$ is not Hopfian

   note: $\mathbb{R}$ is a $\mathbb{Q}$-vector space and has an infinite basis
   project out one factor to get

a surjective map

Are there infinite Hopfian groups?

Cor 10:

$\boxed{\text{a finitely generated free group is Hopfian}}$

Proof:

let $\phi: F \to F$ be a surjective homomorphism with F a free group

proof is by induction on rank of F, $\rho(F)$

rank $F = 1$: $F \cong \mathbb{Z}$

any homomorphism is of the form
$\phi: \mathbb{Z} \to \mathbb{Z} : k \mapsto nk$ for some n

ϕ is surjective $\Leftrightarrow n = \pm 1$

so ϕ an isomorphism

rank$(F) = n > 1$: write $F = \mathbb{Z} * F'$

so $\rho(F') = \rho(F) - 1 > 0$

by Th$^{\underline{m}}$ 7, $F = F_1 * F_2$ and $\phi(F_1) = \mathbb{Z}$
$\phi(F_2) = F'$

note: F_i are free groups

$\rho(F_i) = n_i < \rho(F)$ and

$\rho(F_1) \geq \rho(\mathbb{Z}) = 1$

$$\rho(F_2) \geq \rho(F') = n-1$$

$$\Rightarrow \quad n_1 = 1 \quad \text{and} \quad n_2 = n-1$$

since Cor 8 $\Rightarrow n_1 + n_2 = n$

so $F_1 \cong \mathbb{Z}$ and $F_2 \cong F'$

induction

thus $\left.\begin{array}{l} \phi|_{F_1} : F_1 \to \mathbb{Z} \text{ surjective} \\ \phi|_{F_2} : F_2 \to F' \text{ surjective} \end{array}\right\} \Rightarrow \phi|_{F_1}, \phi|_{F_2}$ are isomorphisms

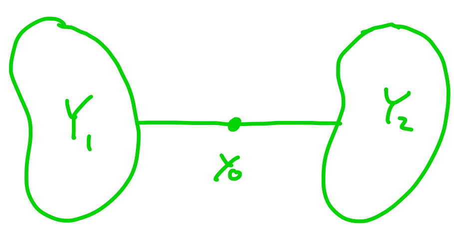
$\therefore$ ϕ an isomorphism

Proof of Theorem 7:

by Theorem 4 we have connected 2-complexes Y_i such that $\pi_1(Y_i) \cong G_i \quad i = 1, 2$

let $Y = Y_1 \cup_h [-1,1] \cup_k Y_2$ where $h : \{-1\} \to Y_1$

$k : \{1\} \to Y_2$



$y_0 = 0 \in [-1, 1]$

to apply Seifert-Van Kampen let

$$A = Y_1 \cup_h [-1, 1) \quad \text{and} \quad B = (-1, 1] \cup_k Y_2$$

exercise: $A \simeq Y_1$ and $B \simeq Y_2$

$$\text{so } \pi_1(A, y_0) \cong G_1 \quad \text{and} \quad \pi_1(B, y_0) \cong G_2$$

now $A \cap B = (-1,1) \simeq pt$ so $\pi_1(A \cap B, y_0) = \{e\}$

so Seifert-Van Kampen $\Rightarrow \pi_1(Y, y_0) \cong G_1 * G_2$

let $\mathcal{P}$ = set of pairs (X, f) such that

1) X connected 2-complex with $\pi_1(X) = F$
2) $f: X \to Y$ such that $f_* = \phi$
3) $f^{-1}(x_0)$ is a finite set of points $(\subset X^{(0)})$

lemma 11:

$$\boxed{\mathcal{P} \neq \emptyset}$$

lemma 12:

$$\boxed{\exists\, (X, f) \in \mathcal{P} \text{ with } f^{-1}(y_0) \text{ a single point}}$$

given 12 we finish the proof of the theorem

let (X, f) be as in lemma 12 and let $x_0 = f^{-1}(y_0)$

let $\{X_\lambda\}$ be the components of $X - \{x_0\}$

(note they are open in X)

set $Y_1^+ = Y_1 \cup_h [-1, 0]$ and $Y_2^+ = [0, 1] \cup_k Y_2$

note: Y_i^+ deformation retracts to Y_i. $i = 1, 2$

and $Y_1^+ \cap Y_2^+ = \{y_0\}$

for each λ, $f(X_\lambda) \subset Y_1^+$ or Y_2^+

let $X_i = \left[\bigcup \{X_\lambda : f(X_\lambda) \subset Y_i^+\} \right] \cup \{x_0\}$

<u>note:</u> $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{x_0\}$

by Seifert-Van Kampen

$$F \cong \pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_2(X_2, x_0)$$

$$\text{set } F_i = \pi_1(X_i, x_0)$$

(technically need X_1, X_2 <u>open</u> to apply Seifert-Van Kampen, can show x_0 has open nbhd in X that is $\simeq$ to x_0 so should use $X_1 \cup N$ and $X_2 \cup N$)

prove this!

note $f(X_i) \subset Y_i^+$ so

$$\underbrace{f_*(\underbrace{\pi_1(X_i, x_0)}_{\cong F_i})}_{\phi(F_i)} \subset \underbrace{\pi_1(Y_i^+, y_0)}_{\cong G_i}$$

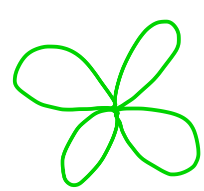
and $\phi|_{F_i} : F_i \to G_i$ is onto since if not then

$$\phi : F \to G_1 * G_2$$

would not be onto

Proof of lemma 11:

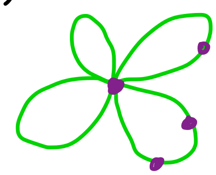
let X = wedge of circles



with $\pi_1(X) = F$

by Theorem 5, $\exists$ a continuous map $f: X \to Y$ such that $f_* = \phi$

Fact: by a small homotopy can arrange $f^{-1}(y_0)$ a finite set of points, then add them to the 0-skeleton $X^{(0)}$



(this is easy to prove with a little differential topology. See Hatcher "Algebraic Topology")

Proof of lemma 12:

let $(X, f) \in \mathcal{P}$ we will modify (X, f) to get the desired pair

first note: $f^{-1}(y_0) \neq \emptyset$

(since if it were empty, then $f(X) \subset Y_1^+$ or Y_2^+ but then $\phi(F) = f_*(\pi_1(X, x_0)) \subset \pi_1(Y_2^+, y_0) = G_2$ $\cancel{\times}$ ϕ onto, unless the other $G_j = \{e\}$ in which case theorem is clearly true!)

suppose $|f^{-1}(y_0)| > 1$

CW Fact: given a map between CW complexes $f: X \to Y$, f may be homotoped so that $f(X^{(2)}) \subset Y^{(2)}$ (2-skeleton

cellular approximation theorem

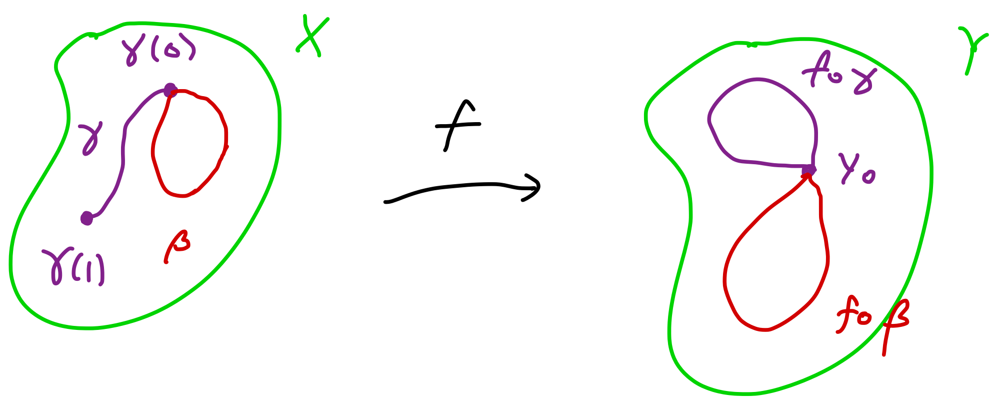
see Hatcher section 4.8

maps to the i-skeleton

let $\gamma : [0,1] \to X$ be a path such that $\gamma(0) \neq \gamma(1)$
and $\gamma(0), \gamma(1) \in f^{-1}(y_0)$

by CW Fact we can assume $\operatorname{im} \gamma \subset X^{(1)}$

note $f \circ \gamma : [0,1] \to Y$ is a loop based at y_0



since $f_* = \phi$ is onto $\exists\, \beta : [0,1] \to X^{(1)} \subset X$ a loop based at $\gamma(0)$ such that $f_*[\beta] = [f \circ \gamma]$

set $\alpha = \bar{\beta} * \gamma$

α is a path in $X^{(1)}$ from $\gamma(0)$ to $\gamma(1)$

such that
$$
\begin{aligned}
[f \circ \alpha] &= [(f \circ \bar{\beta}) * (f \circ \gamma)] \\
&= (f_*([\beta]))^{-1} f_*([\beta]) \\
&= e \text{ in } \pi_1(Y, y_0)
\end{aligned}
$$

let $\alpha = \alpha_1 * \alpha_2 * \dots * \alpha_n$ be a composition of paths $\alpha_i : [0,1] \to X^{(1)} \subset X$ such that

$\alpha_i(\partial I) \subset f^{-1}(y_0)$ and

$f \circ \alpha_i(I) \subset Y^+_{j_i}$ $\quad j_i = 1$ or 2

and $j_i \neq j_{i+1}$

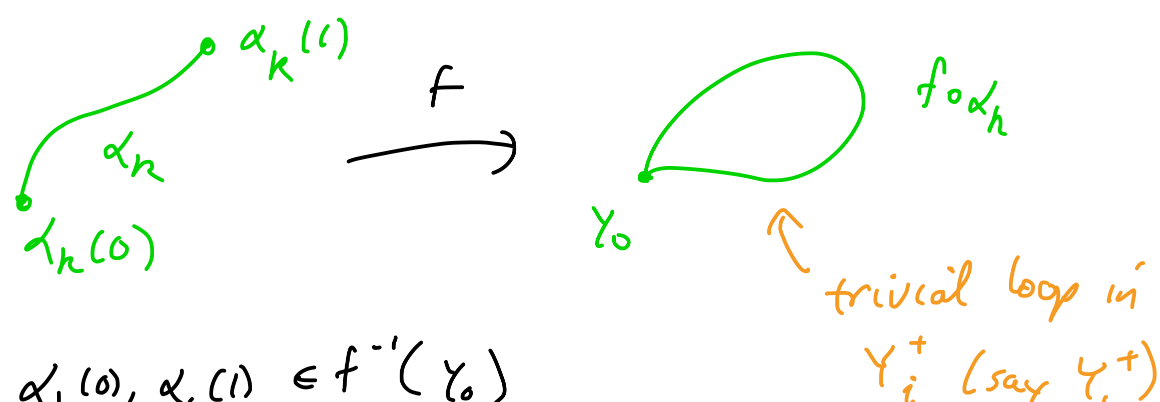
$f \circ \alpha = (f \circ \alpha_1) * \dots * (f \circ \alpha_n)$

composition of loops in Y

so $e = [f \circ \alpha] = [f \circ \alpha_1] \cdot \ldots \cdot [f \circ \alpha_n]$

$= \underbrace{g_1 \cdot g_2 \cdots g_n}_{\substack{\text{word in } G_1 \cup G_2 \\ \text{but not reduced!}}} \in G_1 * G_2$

so must have some k such that $g_k = 1$



$\alpha_k(0), \alpha_k(1) \in f^{-1}(y_0)$

if $\alpha_k(0) = \alpha_k(1)$, then let $\alpha' = \alpha_1 \cdots \hat{\alpha}_k \cdots \alpha_n$

this means remove

exercise: $f \circ \alpha' \simeq f \circ \alpha$

(so $f \circ \alpha' \sim e$ too)

note: we have reduced the length by one

so can't keep doing this at some point

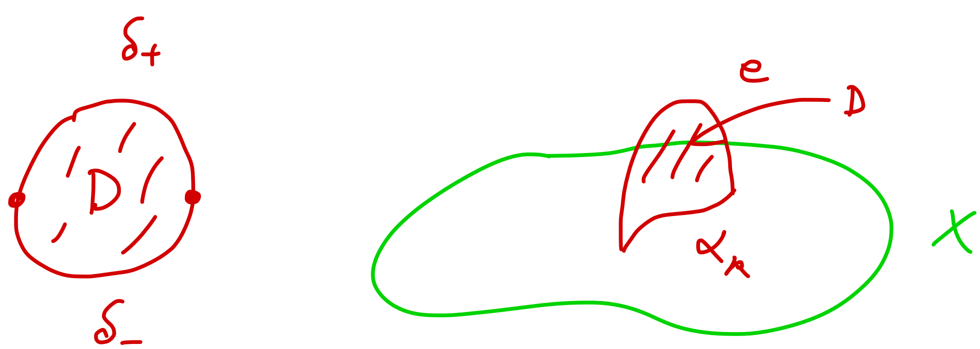
$$\alpha_k(0) \neq \alpha_k(1)$$

let $X' = X \cup \underbrace{(1\text{-cell})}_{e} \cup \underbrace{(2\text{-cell})}_{D}$

glue e to X by $\partial e = \{\alpha_k(0), \alpha_k(1)\}$

let $a : \partial D \to X \cup e$ be given by α_k on δ_-

and homeo. to e on δ_+



note: D deformation retracts to δ_-

exercise: this implies X' deformation retracts to X

(so $X' \simeq X$)

now define $f' : X' \to X$

by $f'|_X = f$ and

$f'|_e$ is the constant map y_0

note $f' \circ a : \partial D \to Y$ is

homotopic to $f \circ \alpha_k \simeq$ constant loop

$\therefore$ can extend $f'|_{\partial D}$ to D

(i.e. all of X')

by the proof of lemma III.14

(see Homework #5)

now

$$\begin{tikzcd} X \arrow[r, phantom, "\overset{\sim}{\subseteq}"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y \end{tikzcd}$$

so $f'_* = f_* = \phi$

<u>note</u>: $f(D) \subset Y_1^+$

<u>Claim</u>: can assume $y_0 \in f(\text{int } D)$

try to show this

so $(f')^{-1}(y_0) = f^{-1}(y_0) \cup e$

let $X'' = X'/e$

we get an induced map $f'': X'' \to Y$

<u>exercise (challenging, we might do later)</u>:

the quotient map $X' \xrightarrow{q} X''$ is a homotopy equivalence since e is homotopic to a point

so $(f'')_* = (f')_* = f_* = \phi$

and $|(f'')^{-1}(y_0)| = |f^{-1}(y_0)| - 1$

so iterating we can eventually get to the point where the preimage of y_0 is one point