C More on Free Products

Thm 10 (Kurosh Subgroup Thm)

Let H be a subgroup of a free product $A*B$

Then $H = (*_\lambda H_\lambda) * F$ where

H_λ is a subgroup of a conjugate of A or B

and F is a free group

Cor 11:

an indecomposable subgroup of a free product is isomorphic to $\mathbb{Z}$ or contained in a conjugate of a factor

Proof: if H is indecomposable then by Thm 10, H is conjugate into a factor or H is a free group, and the only indecomposable free group is $\mathbb{Z}$ ▦

Cor 12:

If two non-trivial elements of a free product commute, then they are either powers of a single element or are both contained in a conjugate of a factor

Proof: suppose $x, y \in A * B$ commute

let $H = \langle x, y \rangle$ = subgroup generated by x and y

then H is abelian, $\therefore$ H indecomposable

so $H \cong \mathbb{Z}$ ($\Rightarrow x, y$ powers of a single element)

or $H \subset$ conjugate of a factor $\blacksquare$

Cor 13:

> If two elements of a free group commute then they are powers of a single element

Cor 14:

> The center of a non-trivial free product is trivial

Proof: suppose $A \neq \{e\} \neq B$

take $a \in A$ and $b \in B$ non-trivial

note:

$$A * B \longrightarrow A * B / N(B) \cong A$$

$$ab \longmapsto a \neq e$$

exercise: show this

normal subgroup generated by B

so $ab \in$ conjugate of B

similarly $ab \in$ conjugate of A

now let $z \in Z(A * B)$

center

suppose $z \neq e$

z and ab commute $\therefore$ by Cor 12

z and ab are powers of a single element (since $ab \notin$ conjugate of a factor)

but ab is not a proper power so

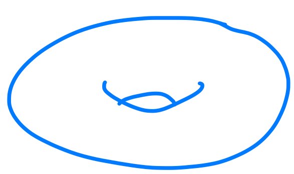
$$z = (ab)^n \quad \text{for some } n > 0$$

note: $a(ab)^n \neq (ab)^n a$

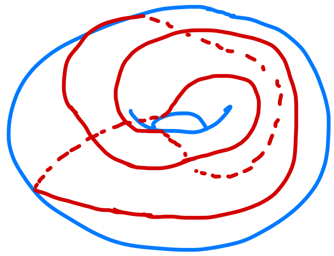
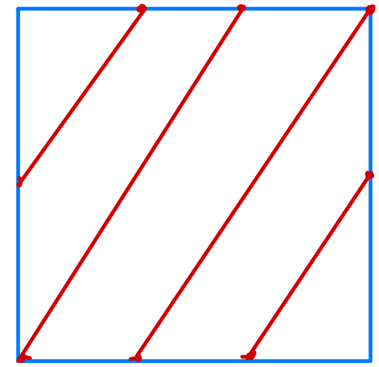
this contradiction $\Rightarrow Z(A * B) = \{e\}$

<u>application to knot theory</u>

let T^2 be the standardly embedded torus in S^3



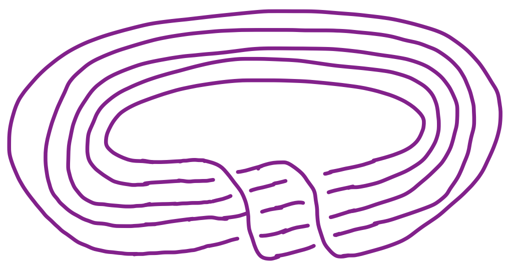
any knot that sits on T^2 is called a <u>torus knot</u>

given any curve of slope q/p, with $(p,q)=1$, in $\mathbb{R}^2$

you get a curve $K_{p,q} \subset \mathbb{R}^2/\mathbb{Z}^2 = T^2$

so you get a (p,q)-torus knot



<u>exercise</u> $\pi_1(X_{K_{p,q}}) \cong \langle x, y \mid x^p y^{-q} \rangle$

<u>Hint:</u> 1) don't use Wirtinger presentation!

2) can write $S^3 = \underbrace{S^1 \times D^2}_{V_1} \cup \underbrace{D^2 \times S^1}_{V_2}$

consider $A = V_1 \cap X_{K_{p,q}}$ and

$B = V_2 \cap X_{K_{p,q}}$

in the Seifert-Van Kampen th$^{\underline{m}}$

Th^{m} 15:

1) $K_{p,q}$ is unknot $\iff$ p or $q = \pm 1$
2) if $p, q, p', q' > 1$, then

$$K_{p,q} \text{ is isotopic to } K_{p',q'}$$
$$\iff$$
$$\{p,q\} = \{p',q'\}$$

Proof:

if p or $q = \pm 1$ then 

and one can easily use Reidemeister type one moves to show this is the unknot

if $p, q > 1$, then let $z = x^p = y^q$

clearly z commutes with x and y

so $z \in Z(\pi_1(X_{K_{p,q}}))$

and so $\langle z \rangle$ is a normal subgroup

$$\pi_1(X_{K_{p,q}}) / \langle z \rangle \cong \langle x, y \mid x^p, y^q \rangle$$
$$\cong \mathbb{Z}_p * \mathbb{Z}_q$$

note: this is non-abelian so $\pi_1(X_{K_{p,q}})$

is non-abelian

but $\pi_1(X_{\text{unknot}}) \cong \mathbb{Z}$ so $K_{p,q}$ is not isotopic to the unknot

now by Cor 14 $Z(\mathbb{Z}_p * \mathbb{Z}_q) = \{e\}$

$\therefore Z(\pi_1(X_{K_{p,q}})) = \langle z \rangle$

exercise

if $G_1 \cong G_2$, then $G_1/Z(G_1) \cong G_2/Z(G_2)$

$\therefore K_{p,q}$ isotopic to $K_{p',q'} \Rightarrow \pi_1(X_{K_{p,q}}) = \pi_1(X_{K_{p',q'}})$

$\Rightarrow \mathbb{Z}_p * \mathbb{Z}_q \cong \mathbb{Z}_{p'} * \mathbb{Z}_{q'}$

Cor 11 $\Rightarrow \mathbb{Z}_p \subset$ conjugate of $\mathbb{Z}_{p'}$ and $\mathbb{Z}_{q'}$

$\therefore p \mid p'$ or $p \mid q'$

similarly $q \mid p'$ or $q \mid q'$ and $p' \mid p$ or $p' \mid q$

and $q' \mid p$ or $q' \mid q$

exercise: 1) this implies $\{p,q\} = \{p',q'\}$

2) if $p = q'$ and $q = p'$, show $K_{p,q}$ is isotopic to $K_{p',q'}$

$\blacksquare$

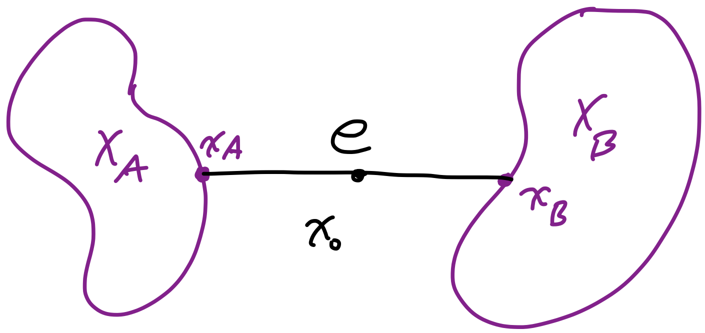
Proof of Th$^{\underline{m}}$ 10 (Kurosh):

let X_A and X_B be connected 2-complexes with

$\pi_1(X_A, x_A) \cong A$ and $\pi_1(X_B, x_B) \cong B$

(can assume x_A, x_B are only 0-cells of X_A and X_B)

let X be obtained from $X_A \amalg X_B$ by attaching a 1-cell e as follows



let $x_0 \in int(e)$

by Seifert-Van Kampen theorem

$$\pi_1(X, x_0) \cong \pi_1(X_A) * \pi_1(X_B)$$

$$\cong A * B$$

careful with base point, eg $\pi_1(X_A, x_A) \cong \pi_1(X_A \cup e, x_0)$

let H be a subgroup $A * B$

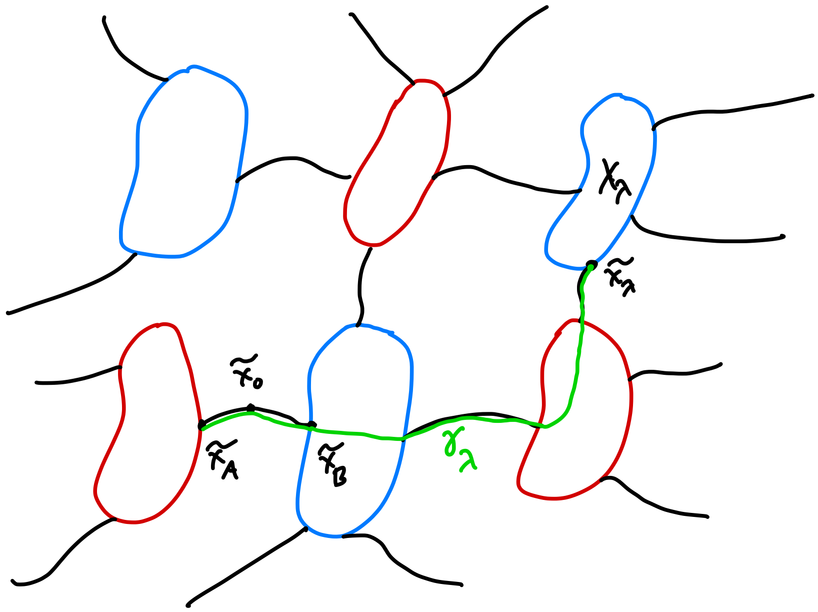
$\exists$ a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$

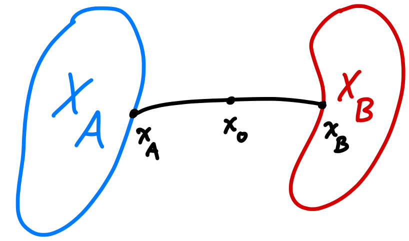
s.t. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$

note: each component of $p^{-1}(X_A)$ is a covering space of X_A

" " $p^{-1}(X_B)$ " " X_B

" " $p^{-1}(e)$ is a 1-cell





let $\tilde{x}_A$ and $\tilde{x}_B$ be end points of 1-cell in $\tilde{X}$ containing $\tilde{x}_0$

let X_λ be a component of $p^{-1}(X_A)$ or $p^{-1}(X_B)$, say $p^{-1}(X_A)$

choose $x_\lambda \in X_\lambda$ s.t. $x_\lambda \in p^{-1}(x_A)$

let γ_λ be a path from x_λ to $\tilde{x}_A$

then $p \circ \gamma_\lambda$ is a loop in X based at x_A

$$\pi_1(X, x_A) \cong \pi_1(X, x_0)$$
$$\begin{array}{c}\rotatebox{90}{$\in$} \\ [p \circ \gamma_\lambda] \longmapsto g_\lambda\end{array}$$

$$\pi_1(X_\lambda, x_\lambda) \to \pi_1(\tilde{X}, x_\lambda) \xrightarrow{\cong} \pi_1(\tilde{X}, \tilde{x}_A) \xrightarrow{p_*} \pi_1(X, x_A) \xrightarrow{\cong} \pi_1(X, x_0)$$
$$\alpha \longmapsto \alpha \longmapsto \bar{\gamma}_\lambda * \alpha * \gamma_\lambda \longmapsto g_\lambda^{-1}[\alpha] g_\lambda$$

this composition takes $\pi_1(X_\lambda, x_\lambda) \subset g_\lambda^{-1} \pi_1(X_A \cup e, x_A)\, g_\lambda$

$$= g_\lambda^{-1} A g_\lambda$$

now let X_λ' be obtained from X_λ by quotienting a maximal tree in $X_\lambda^{(1)}$

just as in the proof of lemma VIII. 8 $X'_\lambda \simeq X_\lambda$

and X'_λ has a single 0-cell x_λ

so let $\tilde{X}'$ be $\tilde{X}$ with all X_λ changed as above

so $\tilde{X}' \simeq X$ and $\tilde{X}'$ = graph Γ with X'_λ's attached at the verticies

now let $\tilde{X}'' = \tilde{X}'$ with a maximal tree of Γ collapsed to a point (still call resulting graph Γ)

as above $\tilde{X}'' \simeq \tilde{X}'$

Seifert-Van Kampen now yields

$$\pi_1(\tilde{X}, \tilde{x}_0) \cong \pi_1(\tilde{X}'', \tilde{x}_0) \cong (*_\lambda \pi_1(X'_\lambda, x'_\lambda)) * \pi_1(\Gamma)$$

$$\cong (*_\lambda \pi_1(X'_\lambda, x'_0)) * \pi_1(\Gamma)$$

from above $p_*(\pi_1(X'_\lambda, x'_0))$ is in a conjugate of A or B

and $\pi_1(\Gamma)$ is a free group

so we see $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has the desired form $\boxplus$