

Math 4441 - Fall 2019 Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 1, 3, 4, 5, 6, 8, 9, 10. **Due: In class September 17**

1. Using the proof of the Fundamental theorem of plane curves (Theorem II.4 in class), find the curve whose signed curvature is 2, passes through the point $(1, 0)$ and whose tangent vector at $(1, 0)$ is $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$.
2. Find a parameterization of a curve in \mathbb{R}^2 with signed curvature is given by $\kappa_\sigma(s) = \frac{1}{1+s^2}$.
3. Is there an isometry of \mathbb{R}^2 taking the curve given by $\alpha(t) = (1 + \cos t, 2 + \sin t)$ for $t \in [0, \pi]$ to the curve given by $\beta(t) = (t, \sin t)$ for $t \in [0, c]$ where c is some constant? Justify your answer.
4. Draw closed plane curves with rotation number (index) equal to 2, 3, -2 and 0.
5. Are the closed curves $\alpha(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ and $\beta(t) = (2 \cos t, -2 \sin t)$ for $t \in [0, 2\pi]$ regular homotopic? Justify your answer.
6. Draw a closed plane regular curve with positive signed curvature that is not convex.
7. Let C be a closed convex regular curve in \mathbb{R}^2 . Show that C has no self-intersections.
8. If you had 6 feet of fencing could you fence of a region of area 3 square feet? Justify your answer. (That is is there a curve in the plane of length 6 feet that bounds a region of area 3 square feet?)
9. Suppose that α is a curve parameterized by arc length and that there is some s_0 such that $\|\alpha(s)\| \leq \|\alpha(s_0)\|$ for all s near s_0 . Show that

$$|\kappa(s_0)| \geq \frac{1}{\|\alpha(s_0)\|}.$$

Hint: The hypothesis says that s_0 is a local maximum of $f(s) = \|\alpha(s)\|^2$. Show that this implies that $\alpha(s_0)$ is a multiple of the normal vector $\hat{N}(s_0)$.

10. Let $\alpha : [0, l] \rightarrow \mathbb{R}^2$ parameterize a simple closed curve by arc length. Suppose that there is a constant R such that the curvature of α satisfies $0 < \kappa_\sigma(s) \leq R$. Prove that

$$\text{length}(\alpha) \geq \frac{2\pi}{R}.$$

Note that the hypothesis say that the curve is curved less than the circle of radius $\frac{1}{R}$, then its length is greater than the length of the circle.

11. Replacing the assumption that α is a simple curve in Problem 10 with the statement that it has rotation number n show that

$$\text{length}(\alpha) \geq \frac{2n\pi}{R}.$$

12. Suppose C_0 and C_1 are two regular curves in \mathbb{R}^2 and they both go through a point \mathbf{p} and are tangent at \mathbf{p} . If near \mathbf{p} the curve C_0 always lies between the tangent line at \mathbf{p} and C_1 then the curvature of C_0 at \mathbf{p} is less than or equal to the curvature of C_1 at \mathbf{p} . Hint: Using a rigid motion you can assume that $\mathbf{p} = (0, 0)$ and the tangent line is the x -axis. Given this we showed in class that C_0 and C_1 can be given, near $(0, 0)$, by the graph of functions f_0 and f_1 , respectively.
13. Let $\alpha : [a, b] \rightarrow \mathbb{R}^3$ be a regular space curve with nonzero curvature at \mathbf{p} . Show that the planar curve obtained by projecting α into its osculating plane at \mathbf{p} has the same curvature at \mathbf{p} as α does at \mathbf{p} . Is this true for any other plane containing the tangent line of α at \mathbf{p} ?