

## Math 4441 - Fall 2019 Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 3, 5, 7, 8, 10, 11. **Due: In class October 22**

1. Compute the first fundamental form for the unit sphere in stereographic coordinates.
2. Let  $\alpha : [a, b] \rightarrow \mathbb{R}^3$  and  $\beta : [a, b] \rightarrow \mathbb{R}^3$  be two regular parameterizations of curves. Define the function

$$\mathbf{f}(u, v) = \alpha(u) + v\beta(u).$$

When does the image of  $\mathbf{f}$  give a regular surface (here do not worry about injectivity of the map, just consider whether the map has derivative of rank 2).

Hint: Recall two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent if and only if  $\mathbf{v}_1 \times \mathbf{v}_2 \neq 0$ .

3. Define the map

$$\pi : (\mathbb{R}^3 - \{(0, 0, 0)\}) \rightarrow S^2$$

by  $\pi(\mathbf{p}) = \frac{\mathbf{p}}{\|\mathbf{p}\|}$ . Show that if  $\Sigma_R$  is the sphere of radius  $R > 0$  about the origin, then the Gauss map of  $\Sigma_R$  is just  $\pi|_{\Sigma_R}$ . Now compute the shape operator and the Gauss curvature of the sphere.

4. Compute the first fundamental form of

$$\mathbf{f}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$$

where  $a, b$  and  $c$  are positive constants.

5. Let  $\Sigma$  be a regular surface in  $\mathbb{R}^3$  with Gauss curvature larger than zero. Given any regular curve  $C$  contained in  $\Sigma$  and point  $\mathbf{p}$  on  $C$ , let  $\kappa_1$  and  $\kappa_2$  be the principal curvatures of  $\Sigma$  at  $\mathbf{p}$  and  $\kappa(\mathbf{p})$  the curvature of  $C$  at  $\mathbf{p}$ . Show that the following inequality is true

$$\kappa(\mathbf{p}) \geq \min\{|\kappa_1|, |\kappa_2|\}.$$

6. The previous problem showed that the principal curvatures could be used to bound the curvature of a curve in a surface from below (at least if the Gauss curvature is positive), can they be used to bound the curvature of a curve in the surface from above? In other words show that if the principal curvatures of  $\Sigma$  are both bounded between  $-c$  and  $c$  then there is no upper bound on what the curvature of a curve in  $\Sigma$  might be.
7. Suppose that  $\mathbf{v}$  and  $\mathbf{w}$  are orthogonal unit vectors in  $T_{\mathbf{p}}\Sigma$ . Show that  $\kappa_{\mathbf{p}}(\mathbf{v}) + \kappa_{\mathbf{p}}(\mathbf{w})$  is independent of the specific choice of  $\mathbf{v}$  and  $\mathbf{w}$  (so long as they are orthogonal).
8. Let  $\alpha(s) = (f(s), g(s))$  be a plane curve parameterized by arc length thought of as sitting in the  $yz$ -plane. Assume that  $f(s) > 0$  for all  $s$ . The surface of revolution  $\Sigma_{\alpha}$  obtained by rotating the curve parameterized by  $\alpha$  about the  $z$ -axis can be parameterized by

$$\mathbf{f}(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$

Compute the unit normal vector and first and second fundamental forms of  $\Sigma_{\alpha}$ .

9. With  $\Sigma_{\alpha}$  as in the previous problem show that the Gauss curvature can be expressed by

$$K(u, v) = -\frac{f''(u)}{f(u)}.$$

Hint: Use the fact that  $\alpha$  is an arc length parameterization.

10. Let

$$\mathbf{f}(u, v) = (u, v, h(u, v))$$

be a parameterization of the graph  $\Gamma_h$  of  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Compute a unit normal vector to  $\Gamma_h$  and the first and second fundamental forms for  $\Gamma_h$ .

11. Considering  $\Gamma_h$  from the previous problem show that the Gauss curvature can be expressed as

$$K(u, v) = \frac{\det(\text{Hess}(h(u, v)))}{(1 + \|\text{grad } h(u, v)\|^2)^2},$$

where  $\text{Hess}(h(u, v))$  is the Hessian of  $h$ , that is the matrix of mixed second partial derivatives of  $h$

$$\text{Hess}(h(x, y)) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(x, y) & \frac{\partial^2 h}{\partial x \partial y}(x, y) \\ \frac{\partial^2 h}{\partial y \partial x}(x, y) & \frac{\partial^2 h}{\partial y^2}(x, y) \end{bmatrix}$$

and  $\text{grad } h$  is the gradient of  $h$ .

12. Compute the Gauss curvature of the graph  $z = ax^2 + by^2$  where  $a$  and  $b$  are constants. (You might want to think a little bit about what the graph looks like for various  $a$  and  $b$  and what the corresponding curvature is.)