Math 4441 - Fall 2020
Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 4, 6, 8, 9, 10.* **Due: In class August 28**

1. Give a formula for the arc length along an ellipse. Look up the integral in a standard table or on-line. What is the integral called? Any guesses as to why? What does this mean concerning parameterizing the ellipse by arc length?

2. Show that if $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is a regular parameterization of a curve then the curvature at $\boldsymbol{\alpha}(t)$ is
$$\kappa(t) = \left\| \left(\frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|} \right)' \frac{1}{\|\boldsymbol{\alpha}'(t)\|} \right\|.$$

3. Let $\boldsymbol{\alpha}(t) = (a\cos^2 t, a\sin t\cos t, a\sin t)$. Find the curvature of $\boldsymbol{\alpha}$.

4. Find the curvature of the ellipse: $\boldsymbol{\alpha}(t) = (a\cos t, b\sin t)$.

5. Give a formula, in terms of $x(t)$ and $y(t)$ and their derivatives, for the curvature for a curve in $\mathbb{R}^2$ given by $\boldsymbol{\alpha}(t) = (x(t), y(t))$.

6. Give a formula, in terms of f and its derivatives, for the curvature of a curve in $\mathbb{R}^3$ given by $\{(x, y, z) : y = x, z = f(x)\}$.

7. Prove that the curvature of the curve in $\mathbb{R}^2$ given as the graph $y = f(x)$ at $(x, f(x))$ is given by
$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

8. Let C be a plane curve parameterized by arc length by $\boldsymbol{\alpha}(s)$, $\boldsymbol{T}(s)$ its unit tangent vector and $\boldsymbol{N}(s)$ be its unit normal vector. Show
$$\frac{d}{ds}\boldsymbol{N}(s) = -\kappa(s)\boldsymbol{T}(s).$$

9. Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ be a parameterized curve in $\mathbb{R}^n$ such that $\boldsymbol{\alpha}(a) = \boldsymbol{p}$ and $\boldsymbol{\alpha}(b) = \boldsymbol{q}$. Show that for any constant vector $\boldsymbol{v}$ with $\|\boldsymbol{v}\| = 1$ we have
$$(\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{v} = \int_a^b \boldsymbol{\alpha}'(t) \cdot \boldsymbol{v}\, dt \leq \int_a^b \|\boldsymbol{\alpha}'(t)\|\, dt.$$
Using this show that
$$\|\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)\| \leq \int_a^b \|\boldsymbol{\alpha}'(t)\|\, dt = \text{lenght}(\boldsymbol{\alpha})$$
Since $\|\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)\|$ is the length of the line segment joining $\boldsymbol{p}$ to $\boldsymbol{q}$ we see that in $\mathbb{R}^n$ the shortest path between two points is a line!
Hint: For the first part recall the Cauchy-Schwarz inequality and for the second part consider $\boldsymbol{v} = \frac{\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)}{\|\boldsymbol{\alpha}(b) - \boldsymbol{\alpha}(a)\|}$.

10. Assume that $\boldsymbol{\alpha}$ is a regular curve in $\mathbb{R}^2$ and all the normal lines of the curve pass though the origin. Prove that $\boldsymbol{\alpha}$ is contained in a circle around the origin. (Recall the normal line at $\boldsymbol{\alpha}(t)$ is the line through $\boldsymbol{\alpha}(t)$ pointing in the direction of the normal vector $\boldsymbol{N}(t)$.)

11. Assume that $\boldsymbol{\alpha}$ is a regular curve in $\mathbb{R}^2$ and all of the tangent lines of $\boldsymbol{\alpha}$ pass through the origin. Show that $\boldsymbol{\alpha}$ is contained in a straight line through the origin. (Recall the tangent line at $\boldsymbol{\alpha}(t)$ is the line through $\boldsymbol{\alpha}(t)$ pointing in the direction of the tangent vector $\boldsymbol{T}(t)$.)