Math 4441 - Fall 2020
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 3, 4, 5, 6, 8, 9, 10.* **Due: September 11**

1. Using the proof of the Fundamental theorem of plane curves (Theorem II.4 in class), find the curve whose signed curvature is 2, passes through the point $(1,0)$ and whose tangent vector at $(1,0)$ is $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$.

2. Find a parameterization of a curve in $\mathbb{R}^2$ with signed curvature is given by $\kappa_\sigma(s) = \frac{1}{1+s^2}$.

3. Is there an isometry of $\mathbb{R}^2$ taking the curve given by $\boldsymbol{\alpha}(t) = (1+\cos t, 2+\sin t)$ for $t \in [0,\pi]$ to the curve given by $\boldsymbol{\beta}(t) = (t, \sin t)$ for $t \in [0,c]$ where c is some constant? Justify your answer.

4. Draw closed plane curves with rotation number (index) equal to 2, 3, -2 and 0.

5. Are the closed curves $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$ for $t \in [0,2\pi]$ and $\boldsymbol{\beta}(t) = (2\cos t, -2\sin t)$ for $t \in [0,2\pi]$ regular homotopic? Justify your answer.

6. Draw a closed plane regular curve with positive signed curvature that is not convex.

7. Let C be a closed convex regular curve in $\mathbb{R}^2$. Show that C has no self-intersections.

8. If you had 6 feet of fencing could you fence of a region of area 3 square feet? Justify your answer. (That is is there a curve in the plane of length 6 feet that bounds a region of area 3 square feet?)

9. Suppose that $\boldsymbol{\alpha}$ is a plane curve parameterized by arc length and that there is some s_0 such that $\|\boldsymbol{\alpha}(s)\| \leq \|\boldsymbol{\alpha}(s_0)\|$ for all s near s_0. Show that
$$|\kappa(s_0)| \geq \frac{1}{\|\boldsymbol{\alpha}(s_0)\|}.$$
Hint: The hypothesis says that s_0 is a local maximum of $f(s) = \|\boldsymbol{\alpha}(s)\|^2$. Show that this implies that $\boldsymbol{\alpha}(s_0)$ is a multiple of the normal vector $\widehat{N}(s_0)$.

10. Let $\boldsymbol{\alpha} : [0,l] \to \mathbb{R}^2$ parameterize a simple closed curve by arc length. Suppose that there is a constant R such that the curvature of $\boldsymbol{\alpha}$ satisfies $0 < \kappa_\sigma(s) \leq R$. Prove that
$$\text{length}(\boldsymbol{\alpha}) \geq \frac{2\pi}{R}.$$
Note that the hypothesis say that the curve is curved less than the circle of radius $\frac{1}{R}$, then its length is greater than the length of the circle.

11. Replacing the assumption that $\boldsymbol{\alpha}$ is a simple curve in Problem 10 with the statement that it has rotation number n show that
$$\text{length}(\boldsymbol{\alpha}) \geq \frac{2n\pi}{R}.$$

12. Suppose C_0 and C_1 are two regular curves in $\mathbb{R}^2$ and they both go through a point $\boldsymbol{p}$ and are tangent at $\boldsymbol{p}$. If near $\boldsymbol{p}$ the curve C_0 always lies between the tangent line at $\boldsymbol{p}$ and C_1 then the curvature of C_0 at $\boldsymbol{p}$ is less than or equal to the curvature of C_1 at $\boldsymbol{p}$. Hint: Using a rigid motion you can assume that $\boldsymbol{p} = (0,0)$ and the tangent line is the x-axis. Given this we showed in class that C_0 and C_1 can be given, near $(0,0)$, by the graph of functions f_0 and f_1, respectively.

13. Let $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^3$ be a regular space curve with nonzero curvature at $\boldsymbol{p}$. Show that the planar curve obtained by projecting $\boldsymbol{\alpha}$ into its osculating plane at $\boldsymbol{p}$ has the same curvature at $\boldsymbol{p}$ as $\boldsymbol{\alpha}$ does at $\boldsymbol{p}$. Is this true for any other plane containing the tangent line of $\boldsymbol{\alpha}$ at $\boldsymbol{p}$?