Math 4441 - Fall 2020
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 4, 5, 8, 10, 11.* **Due: September 25**

1. Let $\boldsymbol{\alpha} : [0, l] \to \mathbb{R}^2$ be an arc length parameterized curve with non-vanishing signed curvature. Define
$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa_\sigma(s)}\widehat{\boldsymbol{N}}(s).$$
(Recall $\widehat{\boldsymbol{N}}(s)$ is the normal vector to $\boldsymbol{\alpha}(s)$ obtained by rotating the tangent vector by $\pi/2$ counterclockwise and $\kappa_\sigma(s)$ is the signed curvature of the curve.) The curve parameterized by $\boldsymbol{\beta}$ is called the *evolute* of $\boldsymbol{\alpha}$. Show that the tangent line to $\boldsymbol{\beta}(s)$ agrees with the normal line to $\boldsymbol{\alpha}(s)$.

2. Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^3$ be a regular space curve. Show that you can compute the curvature by the formula
$$\kappa(t) = \frac{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3}$$

3. Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^3$ be a regular space curve. Show that you can compute the torsion by the formula
$$\tau(t) = \frac{(\boldsymbol{\alpha}''(t) \times \boldsymbol{\alpha}'''(t)) \cdot \boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t) \times \boldsymbol{\alpha}''(t)\|^2}.$$

4. Compute the unit tangent vector, unit normal vector, binormal vector, curvature and torsion of the following curve
$$\boldsymbol{\alpha}(t) = (e^t \cos t, e^t \sin t, e^t)$$

5. Show that for any arc length parameterized curve there is a vector $\boldsymbol{\omega}(s)$ that satisfies
$$\boldsymbol{T}'(s) = \boldsymbol{\omega}(s) \times \boldsymbol{T}(s)$$
$$\boldsymbol{N}'(s) = \boldsymbol{\omega}(s) \times \boldsymbol{N}(s)$$
$$\boldsymbol{B}'(s) = \boldsymbol{\omega}(s) \times \boldsymbol{B}(s).$$
HINT: Consider $\boldsymbol{\omega}(s) = a(s)\boldsymbol{T}(s) + b(s)\boldsymbol{N}(s) + c(s)\boldsymbol{B}(s)$ (where $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ are the unit tangent, normal and binormal vectors) and find the coefficients a, b, c that work.

6. Show that if $\boldsymbol{\alpha}$ is a parameterization by arc length of a curve lying on a sphere of radius R about the origin in $\mathbb{R}^3$ then
$$R^2 = \left(\frac{1}{\kappa(s)}\right)^2 + \left(\left(\frac{1}{\kappa(s)}\right)' \frac{1}{\tau(s)}\right)^2.$$
HINT: Write $\boldsymbol{\alpha}(s) = a(s)\boldsymbol{T}(s) + b(s)\boldsymbol{N}(s) + c(s)\boldsymbol{B}(s)$ (where $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ are the unit tangent, normal and binormal vectors) and try to determine what a, b, c are. To do this consider $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} = R^2$. Differentiate this several time and see what relations you get.

7. Show that if a curve lies on a sphere of radius R then then is curvature satisfies

$$\kappa(s) \geq \frac{1}{R}.$$

Hint: For an arc length parameterization $\boldsymbol{\alpha}$ consider the equation $\|\boldsymbol{\alpha}(s) - \boldsymbol{p}\|^2 = R^2$, for some constant vector $\boldsymbol{p}$, and differentiate many times.

8. Suppose a curve has non-zero curvature $\kappa(s)$ and torsion $\tau(s)$. Show that if the curve lies on a sphere of any radius about any point then

$$\frac{\tau(s)}{\kappa(s)} = \left(\frac{\kappa'(s)}{\tau(s)\kappa^2(s)}\right)'.$$

HINT: For an arc length parameterization $\boldsymbol{\alpha}$ consider the equation $\|\boldsymbol{\alpha}(s) - \boldsymbol{p}\|^2 = R^2$, for some constant vector $\boldsymbol{p}$, and differentiate many times.

9. Suppose that $\boldsymbol{\alpha}(s)$ is an arc length parameterization of a curve with non-zero curvature $\kappa(s)$ and torsion $\tau(s)$ that satisfy the equation in the previous problem. Show that the curve lies on a sphere.

   HINT: Consider the point $\boldsymbol{p}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{N} + \left(\frac{1}{\kappa(s)}\right)' \frac{1}{\tau(s)}\boldsymbol{B}$ and show it is constant by taking its derivative. (As always $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ are the unit tangent, normal and binormal vectors.) Now who that $\|\boldsymbol{\alpha}(s) - \boldsymbol{p}\|$ is constant.

10. Let $\boldsymbol{\alpha}(s)$ and $\boldsymbol{\beta}(s)$ be two unit speed curves. Assume $\kappa_{\boldsymbol{\alpha}}(s) = \kappa_{\boldsymbol{\beta}}(s)$ and $\tau_{\boldsymbol{\alpha}}(s) = \tau_{\boldsymbol{\beta}}(s)$ (where $\kappa_{\boldsymbol{\alpha}}(s)$ refers to the curvature of $\boldsymbol{\alpha}$ and similarly for the other terms). Let

$$D(s) = \boldsymbol{T}_{\boldsymbol{\alpha}}(s) \cdot \boldsymbol{T}_{\boldsymbol{\beta}}(s) + \boldsymbol{N}_{\boldsymbol{\alpha}}(s) \cdot \boldsymbol{N}_{\boldsymbol{\beta}}(s) + \boldsymbol{B}_{\boldsymbol{\alpha}}(s) \cdot \boldsymbol{B}_{\boldsymbol{\beta}}(s).$$

   Recall at the beginning of class you had an exercise that showed that there is an isometry $A : \mathbb{R}^3 \to \mathbb{R}^3$ taking $\boldsymbol{\alpha}(0)$ to $\boldsymbol{\beta}(0)$ and the orthonormal basis $\{\boldsymbol{T}_{\boldsymbol{\alpha}}(0), \boldsymbol{N}_{\boldsymbol{\alpha}}(0), \boldsymbol{B}_{\boldsymbol{\alpha}}(0)\}$ to the orthonormal basis $\{\boldsymbol{T}_{\boldsymbol{\beta}}(0), \boldsymbol{N}_{\boldsymbol{\beta}}(0), \boldsymbol{B}_{\boldsymbol{\beta}}(0)\}$. From now on we will work with the curve $A(\boldsymbol{\alpha})$ but to avoid too much notation we will still call it $\boldsymbol{\alpha}$. So at this point, after applying an isometry, we are assuming $\boldsymbol{\alpha}(0) = \boldsymbol{\beta}(0)$ and $\{\boldsymbol{T}_{\boldsymbol{\alpha}}(0), \boldsymbol{N}_{\boldsymbol{\alpha}}(0), \boldsymbol{B}_{\boldsymbol{\alpha}}(0)\}$ is equal to $\{\boldsymbol{T}_{\boldsymbol{\beta}}(0), \boldsymbol{N}_{\boldsymbol{\beta}}(0), \boldsymbol{B}_{\boldsymbol{\beta}}(0)\}$.

   Without using the fundamental theory of curves show that

   (a) $D(0) = 3$.

   (b) $D(s) = 3$ implies the Frenet frames of α and β agree at s. (Recall the Frenet frame of $\boldsymbol{\alpha}(s)$ is just $\{\boldsymbol{T}_{\boldsymbol{\alpha}}(s), \boldsymbol{N}_{\boldsymbol{\alpha}}(s), \boldsymbol{B}_{\boldsymbol{\alpha}}(s)\}$ and similarly for $\boldsymbol{\beta}(s)$.)

   (c) $D'(s) = 0$ (and thus $D(s) = 3$).

   (d) $\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(s)$ for all s.

   Note you have just shown that a (biregular) curve in $\mathbb{R}^3$ is uniquely determined (up to translation and rotation) by its curvature and torsion.

11. Consider the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. In this problem you will construct steriographic coordinates for $S^2 - \{(0, 0, 1)\}$.

   (a) Given any point $(x, y, 0)$ in the xy-plane, parameterize the line that contains $(x, y, 0)$ and $(0, 0, 1)$.

(b) Show this line intersects S^2 in exactly two points $(0,0,1)$ and another point.

(c) Show that the second point can be expressed as

$$\boldsymbol{f}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

(d) Show that $\boldsymbol{f}$ is a local parameterization of S^2 for every point in S^2 except $(0,0,1)$.