Math 4441 - Fall 2020
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 3, 5, 7, 8, 10, 11.* **Due: October 9**

1. Compute the first fundamental form for the unit sphere in steriographic coordinates.

2. Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^3$ and $\boldsymbol{\beta} : [a, b] \to \mathbb{R}^3$ be two regular parameterizations of curves. Define the function
$$\boldsymbol{f}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u).$$
When does the image of $\boldsymbol{f}$ give a regular surface (here do not worry about injectivity of the map, just consider whether the map has derivative of rank 2).
Hint: Recall two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent if and only if $\boldsymbol{v}_1 \times \boldsymbol{v}_2 \neq 0$.

3. Define the map
$$\boldsymbol{\pi} : (\mathbb{R}^3 - \{(0, 0, 0)\}) \to S^2$$
by $\boldsymbol{\pi}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$. Show that if Σ_R is the sphere of radius $R > 0$ about the origin, then the Gauss map of Σ_R is just $\boldsymbol{\pi}|_{\Sigma_R}$. Now compute the shape operator and the Gauss curvature of the sphere.

4. Compute the first fundamental form of
$$\boldsymbol{f}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$$
where a, b and c are positive constants.

5. Let Σ be a regular surface in $\mathbb{R}^3$ with Gauss curvature larger than zero. Given any regular curve C contained in Σ and point $\boldsymbol{p}$ on C, let κ_1 and κ_2 be the principal curvatures of Σ at p and $\kappa(\boldsymbol{p})$ the curvature of C at $\boldsymbol{p}$. Show that the following inequality is true
$$\kappa(\boldsymbol{p}) \geq \min\{|\kappa_1|, |\kappa_2|\}.$$

6. The previous problem showed that the principal curvatures could be used to bound the curvature of a curve in a surface from below (at least if the Gauss curvature is positive), can they be used to bound the curvature of a curve in the surface from above? In other words show that if the principal curvatures of Σ are both bounded between between $-c$ and c then there is no upper bound on what the curvature of a curve in Σ might be.

7. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal unit vectors in $T_{\boldsymbol{p}}\Sigma$. Show that $\kappa_{\boldsymbol{p}}(\boldsymbol{v}) + \kappa_{\boldsymbol{p}}(\boldsymbol{w})$ is independent of the specific choice of $\boldsymbol{v}$ and $\boldsymbol{w}$ (so long as they are orthogonal).

8. Let $\boldsymbol{\alpha}(s) = (f(s), g(s))$ be a plane curve parameterized by arc length thought of as sitting in the yz-plane. Assume that $f(s) > 0$ for all s. The surface of revolution $\Sigma_{\boldsymbol{\alpha}}$ obtained by rotating the curve parameterized by $\boldsymbol{\alpha}$ about the z-axis can be parameterized by
$$\boldsymbol{f}(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).$$
Compute the unit normal vector and first and second fundamental forms of $\Sigma_{\boldsymbol{\alpha}}$.

9. With $\Sigma_{\boldsymbol{\alpha}}$ as in the previous problem show that the Gauss curvature can be expressed by

$$K(u,v) = -\frac{f''(u)}{f(u)}.$$

Hint: Use the fact that $\boldsymbol{\alpha}$ is an arc length parameterization.

10. Let

$$\boldsymbol{f}(u,v) = (u, v, h(u,v))$$

be a parameterization of the graph Γ_h of $h : \mathbb{R}^2 \to \mathbb{R}$. Compute a unit normal vector to Γ_h and the first and second fundamental forms for Γ_h.

11. Considering Γ_h from the previous problem show that the Gauss curvature can be expressed as

$$K(u,v) = \frac{\det\left(\text{Hess}(h(u,v))\right)}{\left(1 + \|\text{grad } h(u,v)\|^2\right)^2},$$

where $\text{Hess}(h(u,v))$ is the Hessian of h, that is the matrix of mixed second partial derivatives of h

$$\text{Hess}(h(x,y)) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(x,y) & \frac{\partial^2 h}{\partial x \partial y}(x,y) \\ \frac{\partial^2 h}{\partial y \partial x}(x,y) & \frac{\partial^2 h}{\partial y^2}(x,y) \end{bmatrix}$$

and grad h is the gradient of h.

12. Compute the Gauss curvature of the graph $z = ax^2 + by^2$ where a and b are constants. (You might want to think a little bit about what the graph looks like for various a and b and what the corresponding curvature is.)