Math 4441 - Fall 2020
Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 4, 5, 6.* **Due: November 6.**

1. Calculate the Christoffel symbols of the surface parameterized by

$$\boldsymbol{f}(u,v) = (u\cos v, u\sin v, u)$$

by using the definition of Christoffel symbols (that is they are some of the coefficients used to express the vectors $\boldsymbol{f}_{uu}$, $\boldsymbol{f}_{uv}$ and $\boldsymbol{f}_{vv}$ expressed in terms of the tangent vectors $\boldsymbol{f}_u$ and $\boldsymbol{f}_v$ and the normal vector N).

2. Calculate the Christoffel symbols for the surface in the previous problem using the formula that expresses them in terms of the 1st fundamental form.

3. Compute the Gauss curvature of the following Riemannian metrics.

   (a) (The spherical metric)

$$g = \frac{4}{(1+u^2+v^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

   on $\mathbb{R}^2$.

   (b) (The hyperbolic metric)

$$g = \frac{4}{(1-(u^2+v^2))^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

   on the (open) unit disk $D^2 = \{(u,v) \in \mathbb{R}^2 | u^2+v^2 < 1\}$.

   (c) (Hyperbolic metric again)

$$g = \frac{1}{v^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

   on the upper half plane $H = \{(u,v) \in \mathbb{R}^2 | v > 0\}$.

4. Using the Riemannian metric in 3b compute:

   (a) The area of $D_\rho = \{(u,v) | u^2+v^2 \le \rho^2\}$.

   (b) The length of the curve ∂D_ρ.

   (c) Make the same computations using the metric from 3a.

5. Suppose that the 1st fundamental from of a surface (or if you prefer the Riemannian metric for the surface) is given by

$$\begin{bmatrix} \lambda(u,v) & 0 \\ 0 & \lambda(u,v) \end{bmatrix}$$

for some function positive $\lambda(u,v)$. Show that the Gauss curvature of the surface is given by the formula

$$K(u,v) = -\frac{1}{2\lambda}\nabla^2(\ln\lambda)$$

where $\nabla^2 f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$ for any function f.

6. Compute the total curvature of the the unit sphere S^2 in $\mathbb{R}^3$, that is compute

$$\int_{S^2} K(p)dA$$

where $K(p)$ is the Gauss curvature. Do not use the Gauss-Bonnet theorem. In a coordinate chart with coordinates (u,v), $dA = \sqrt{\det g}\, dudv$.
Hint: If you use stereographic coordinates you can cover S^2 minus a point with one coordinate chart, and therefore do all your computation in this coordinate chart. Recall the first fundamental form in stereographic coordinates is given in 3a.

7. Let $\boldsymbol{\alpha} : (-\epsilon,\epsilon) \to \Sigma$ be a regular arc in Σ. For $t \in (-\epsilon,\epsilon)$ let $\boldsymbol{w}(t) \in T_{\boldsymbol{\alpha}(t)}\Sigma$, so $\boldsymbol{w}$ is a vector field along $\boldsymbol{\alpha}$. Define

$$\frac{D\boldsymbol{w}}{dt}(t) = (\nabla_{\boldsymbol{\alpha}'(t)}\boldsymbol{w})(t).$$

If $\boldsymbol{w}$ and $\boldsymbol{v}$ are two vector fields along $\boldsymbol{\alpha}$ show

$$\frac{d}{dt}g\left(\boldsymbol{v}(t),\boldsymbol{w}(t)\right) = g\left(\frac{D\boldsymbol{v}}{dt}(t),\boldsymbol{w}(t)\right) + g\left(\boldsymbol{v}(t),\frac{D\boldsymbol{w}}{dt}(t)\right).$$

8. Let Σ be a surface in $\mathbb{R}^3$ and give Σ the Riemannian metric induced from $\mathbb{R}^3$ (that is the Riemannian metric is the 1st fundamental form).

   (a) Show that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ and $F(\Sigma) = \Sigma$, then F restricted to Σ, denoted $F|_\Sigma : \Sigma \to \Sigma$, is an isometry of Σ.

   (b) Show that $O(3)$, the group of orthogonal linear transformations of $\mathbb{R}^3$, is a subset of the group of isometries of S^2 (the unit sphere). Recall $O(3)$, the group of orthogonal linear transformations of $\mathbb{R}^3$, is the set of 3×3 matrices with determinant ± 1.