Math 4441 - Fall 2020
Extra Practice Problems

1. Let Σ be a regular, connected, compact, orientable surface in $\mathbb{R}^3$ which is not homeomorphic to a sphere. Prove that there are points on Σ where the Gaussian curvature is positive, negative and zero.

2. Determine the Christoffel symbols of a surface represented in the form $z = f(x, y)$.

3. Write down the differential equations that determine a geodesic on the surface given by $z = f(x, y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is any function. That is if $\boldsymbol{\alpha}(t) = \boldsymbol{f}(a(t), b(t))$ is a geodesic. What equations must a and b satisfy?

4. Can there be a smooth closed geodesic curve bounding a disk on a surface with Gauss curvature is (a) strictly positive? (b) strictly negative? (c) zero? Prove your answer.

5. Let $p \in \Sigma$ and $S_r(\boldsymbol{p})$ be the geodesic circle with center $\boldsymbol{p}$ and radius r. Let L be the length of $S_r(\boldsymbol{p})$ and A the area of the region bounded by $S_r(\boldsymbol{p})$. Prove that

$$4\pi A - L^2 = \pi^2 K(\boldsymbol{p}) r^4 + R$$

   where R is a function of r satisfying

$$\lim_{r \to 0} \frac{R}{r^4} = 0.$$

6. Let Σ be the surface parameterized by $\boldsymbol{f}(u, v) = (u \cos v, u \sin v, u^2)$ for $u \geq 0$ and $0 \leq v \leq 2\pi$. Let Σ_r be the portion of the surface with $0 \leq u \leq r$.

   (a) Calculate the geodesic curvature of the boundary circles of Σ_r and also compute $\int_{\partial \Sigma_r} \kappa_g(s)\, ds$.

   (b) What is $\chi(\Sigma_r)$?

   (c) Use the Gauss-Bonnet Theorem to compute $\int_{\Sigma_r} K\, dA$. Compute the limit as $r \to \infty$.

   (d) Compute K directly.

   (e) Use the previous computation to explicitly compute $\int_{\Sigma_r} K\, dA$.