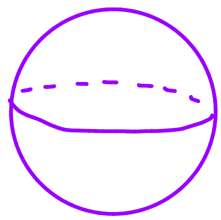
Differential Geometry
- curves and surfaces -

I. Introduction

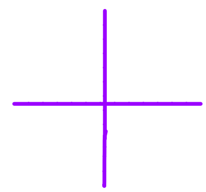
A. What is Differential Geometry

How can you tell if you "live" on the surface of a ball



called a sphere
or 2-sphere

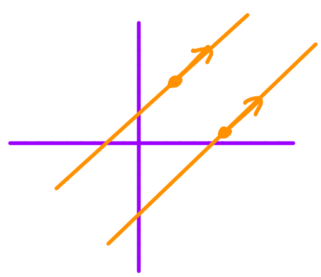
or the plane



?

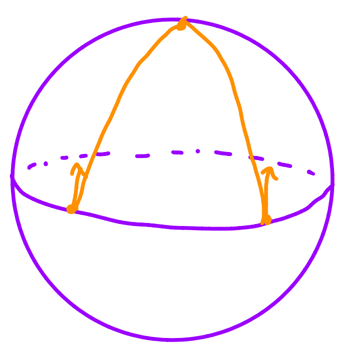
one way is to look at "straight lines"

in the plane



if two people walk in the "same direction" from different points they stay a fixed distance appart
(parallel lines don't intersect)

but on the sphere



the distances get closer together

that is, the "geometry" of "lines" on the sphere is different from the geometry of the plane.

so we see the "curvature" of the sphere by looking at straight lines in the space

Question: What about the 3-dimensional space in which we live?

is it "flat Euclidean space"?

is it a "3-dimensional sphere"?

something else?

General Relativity postulates that gravity can be undestood as a "curvature" in space (time).

The language to study all these ideas is <u>Riemannian Geometry</u>

or more generally <u>Differential Geometry</u>

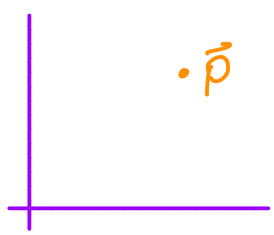
and it all starts with studying <u>curves</u> and <u>straight lines</u>

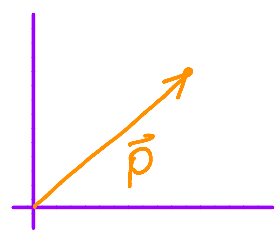
This course is an introduction to Riemannian Geometry through curves and surfaces in Euclidean space (see list of topics on the web page)

B. The geometry of Euclidean Space

$\mathbb{R}^n = \{(p_1, \dots, p_n) \mid p_i$ a real number, i.e. $p_i \in \mathbb{R}\}$

we can think of $\vec{p} = (p_1, \dots, p_n)$ as a <u>point</u> in $\mathbb{R}^n$ or a <u>vector</u> in $\mathbb{R}^n$





when thinking of $\vec{p}$ as a vector we will frequently write it as a column vector

$$\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

given $\vec{p}, \vec{q} \in \mathbb{R}^n$ then their <u>dot product</u> is

$$\vec{p} \cdot \vec{q} = p_1 q_1 + p_2 q_2 + \dots + p_n q_n = \sum_{i=1}^{n} p_i q_i$$

we sometimes write $\langle \vec{p},\vec{q}\rangle$ for $\vec{p}\cdot\vec{q}$ and this gives an inner product on $\mathbb{R}^n$, that is $\langle\cdot,\cdot\rangle$ satisfies

1) $\langle \vec{p},\vec{q}\rangle = \langle \vec{q},\vec{p}\rangle$ — symmetric

2) $\left.\begin{array}{l}\langle a\vec{p},\vec{q}\rangle = a\langle \vec{p},\vec{q}\rangle = \langle \vec{p},a\vec{q}\rangle \\ \langle \vec{p}+\vec{q},\vec{r}\rangle = \langle \vec{p},\vec{r}\rangle + \langle \vec{q},\vec{r}\rangle\end{array}\right\}$ linear

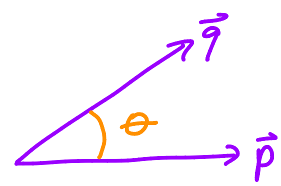
3) $\left.\begin{array}{l}\langle \vec{p},\vec{p}\rangle \geq 0 \\ \langle \vec{p},\vec{p}\rangle = 0 \iff \vec{p}=\vec{0}\end{array}\right\}$ positive definite

geometry is about lengths and angles, with a dot product we can define the length of $\vec{p}$ to be

$$\|\vec{p}\| = \sqrt{\langle \vec{p},\vec{p}\rangle}$$

and the angle between $\vec{p}$ and $\vec{q}$ to be

$$\cos\theta = \frac{\langle \vec{p},\vec{q}\rangle}{\|\vec{p}\|\,\|\vec{q}\|}$$



note for this to be well-defined, we need

lemma 1 (Cauchy-Schwartz inequality):

for all $\vec{p},\vec{q}\in\mathbb{R}^n$

$$|\langle \vec{p},\vec{q}\rangle| \leq \|\vec{p}\|\,\|\vec{q}\|$$

with equality if and only if $\vec{p}$ and $\vec{q}$ are linearly dependent

Proof: nice trick: compute the length of a linear combination

$$0 \leq \|a\vec{p}+b\vec{q}\|^2 = \langle a\vec{p}+b\vec{q}, a\vec{p}+b\vec{q}\rangle$$

$$= a^2\|p\|^2 + b^2\|\vec{q}\|^2 + 2ab\langle \vec{p},\vec{q}\rangle$$

so if $a=\|\vec{q}\|$ and $b=\pm\|\vec{p}\|$, then we have

$$\left.\begin{array}{l} 0 \leq 2\|\vec{p}\|^2\|\vec{q}\|^2 \pm 2\|\vec{p}\|\|\vec{q}\|\langle \vec{p},\vec{q}\rangle \\ = 2\|\vec{p}\|\,\|\vec{q}\|\left(\|\vec{p}\|\,\|\vec{q}\| \pm \langle \vec{p},\vec{q}\rangle\right)\end{array}\right\} (*)$$

so if $\|\vec{p}\| \neq 0 \neq \|\vec{q}\|$, then

$$\pm \langle \vec{p}, \vec{q} \rangle \leq \|\vec{p}\| \|\vec{q}\|$$

(if either $\|\vec{p}\| = 0$ or $\|\vec{q}\| = 0$ then $\leq$ is obvious)

and $\|\vec{p}\| \|\vec{q}\| \geq \max \{ \langle \vec{p}, \vec{q} \rangle, -\langle \vec{p}, \vec{q} \rangle \}$

$$= | \langle \vec{p}, \vec{q} \rangle |$$

thus the $\leq$ in the lemma is true

<u>note</u>: assuming $\|\vec{p}\| \neq 0 \neq \|\vec{q}\|$ then

$$\langle \vec{p}, \vec{q} \rangle = \|\vec{p}\| \|\vec{q}\|$$

$$\Longleftrightarrow$$

we have equality in ⊛

$$\Longleftrightarrow$$

$$\big\| \|\vec{p}\| \vec{q} \pm \|\vec{q}\| \vec{p} \big\| = 0$$

non-degeneracy of inner product

$$\Longleftrightarrow$$

$$\underbrace{\|\vec{p}\| \vec{q} \pm \|\vec{q}\| \vec{p} = 0}$$

i.e. $\vec{p}$ and $\vec{q}$ are linearly dependent

The standard distance between points in $\mathbb{R}^n$ is

$$d(\vec{p}, \vec{q}) = \|\vec{p} - \vec{q}\|$$

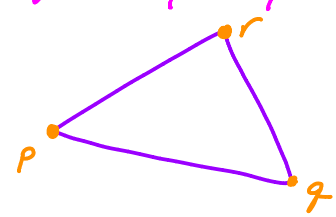
a <u>metric</u> on a set X is a function

$$d : X \times X \to \mathbb{R}$$

(metrics describe distance between points)

such that

1) $d(p,q) \geq 0$ with equality $\Leftrightarrow p = q$
2) $d(p,q) = d(q,p)$
3) $d(p,q) \leq d(p,r) + d(r,q)$ triangle inequality



<u>exercise</u>: Show that $d(\vec{p}, \vec{q})$ above is a metric on $\mathbb{R}^n$

given two metric spaces (M_1, d_1) and (M_2, d_2) an isometry is a surjective function

$$\phi: M_1 \to M_2$$

such that

$$d_2(\phi(x), \phi(y)) = d_1(x,y) \quad \text{for all } x,y \in M_1$$

Isometries identify points of M_1 with points of M_2 so that distances are preserved. They are "symmetries" of spaces with metrics

We are interested in isometries from $(\mathbb{R}^n, d)$ to itself

Notice any "geometric quantity" should not change under isometries (e.g. length of a curve ...)

An orthogonal transform is a linear map

$$A: \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\langle A\vec{p}, A\vec{q} \rangle = \langle \vec{p}, \vec{q} \rangle \quad \text{for all } \vec{p}, \vec{q}$$

Theorem 2:

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then there is some $\vec{a} \in \mathbb{R}^n$ and orthogonal transform A such that

$$f(\vec{p}) = \vec{a} + A\vec{p}$$

Proof: let $\tilde{f}(\vec{p}) = f(\vec{p}) - f(\vec{0})$

if we show $\tilde{f}$ is ① linear and

② satisfies $\langle \tilde{f}(\vec{p}), \tilde{f}(\vec{q}) \rangle = \langle \vec{p}, \vec{q} \rangle$

then we are done since we can set $A = \tilde{f}$ and $\vec{a} = f(\vec{0})$

to get $f(\vec{p}) = A\vec{p} + \vec{a}$

note: $\langle \vec{x}-\vec{y}, \vec{x}-\vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\langle \vec{x}, \vec{y} \rangle$

so

$$2\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2$$

thus

$$\begin{aligned}
2\langle \tilde{f}(\vec{p}), \tilde{f}(\vec{q}) \rangle &= \|\tilde{f}(\vec{p})\|^2 + \|\tilde{f}(\vec{q})\|^2 - \|\tilde{f}(\vec{p}) - \tilde{f}(\vec{q})\|^2 \\
&= \|f(\vec{p}) - f(\vec{0})\| + \|f(\vec{q}) - f(\vec{0})\|^2 - \|f(\vec{p}) - f(\vec{q})\|^2 \\
&\overset{\text{isometry}}{=} \|\vec{p} - \vec{0}\|^2 + \|\vec{q} - \vec{0}\|^2 - \|\vec{p} - \vec{q}\|^2 \\
&= \|\vec{p}\|^2 + \|\vec{q}\|^2 - \|\vec{p} - \vec{q}\|^2 = 2\langle \vec{p}, \vec{q} \rangle
\end{aligned}$$

so $\tilde{f}$ satisfies ②

now let $\vec{e}_1, \dots, \vec{e}_n$ be an orthonormal basis for $\mathbb{R}^n$

$$\left(\text{e.g. } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \dots \right)$$

exercise: $\tilde{f}(\vec{e}_1), \dots, \tilde{f}(\vec{e}_n)$ is also an orthonormal basis for $\mathbb{R}^n$ because of ②

so for any i

$$\begin{aligned}
\langle \tilde{f}(\vec{p}+\vec{q}), \tilde{f}(\vec{e}_i) \rangle &= \langle \vec{p}+\vec{q}, \vec{e}_i \rangle = \langle \vec{p}, \vec{e}_i \rangle + \langle \vec{q}, \vec{e}_i \rangle \\
&= \langle \tilde{f}(\vec{p}), \tilde{f}(\vec{e}_i) \rangle + \langle \tilde{f}(\vec{q}), \tilde{f}(\vec{e}_i) \rangle \\
&= \langle \tilde{f}(\vec{p}) + \tilde{f}(\vec{q}), \tilde{f}(\vec{e}_i) \rangle \quad \text{for all } i
\end{aligned}$$

and thus $\tilde{f}(\vec{p}+\vec{q}) = \tilde{f}(\vec{p}) + \tilde{f}(\vec{q})$

exercise: Prove this if it is not clear to you

Hint: $\vec{b}_1 \dots \vec{b}_n$ an orthonormal basis, then
$\vec{v} = \vec{w} \iff \langle \vec{v}, \vec{b}_i \rangle = \langle \vec{w}, \vec{b}_i \rangle$ for all i

similarly

$$\begin{aligned}
\langle \tilde{f}(c\vec{p}), \tilde{f}(\vec{e}_i) \rangle &= \langle c\vec{p}, \vec{e}_i \rangle = c\langle \vec{p}, \vec{e}_i \rangle \\
&= c\langle \tilde{f}(\vec{p}), \tilde{f}(\vec{e}_i) \rangle = \langle c\tilde{f}(\vec{p}), \tilde{f}(\vec{e}_i) \rangle
\end{aligned}$$

so $\tilde{f}(c\vec{p}) = c\tilde{f}(\vec{p})$

and thus $\tilde{f}$ is linear

So any isometry of $\mathbb{R}^n$ (also called a rigid motion) is a composition of

① an orthogonal transformation

$$f(\vec{p}) = A\vec{p} \qquad \text{and}$$

② a translation

$$f(\vec{p}) = \vec{p} + \vec{a}$$

we understand ②. let's explore ①

Recall: given a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$

we can express it as an $n \times n$ matrix

e.g. let $\vec{e}_1, \dots \vec{e}_n$ be the standard basis for $\mathbb{R}^n$

$$A\vec{e}_i = a_{1i}\vec{e}_1 + \dots + a_{ni}\vec{e}_n$$

let $$M_A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

any vector can be written

$$\vec{v} = v_1\vec{e}_1 + \dots + v_n\vec{e}_n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then $A\vec{v}$ corresponds to the vector $M_A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

from now on we will think of A as the matrix above that represets it in this basis

now with $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

note

$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$ where $\vec{v}^T$ means the transpose of v

ie switch rows and columns

for any matrix A

$$\langle A^T\vec{v}, \vec{w}\rangle = (A^T\vec{v})^T\vec{w} = \vec{v}^T(A^T)^T\vec{w} = \vec{v}^TA\vec{w} = \langle\vec{v}, A\vec{w}\rangle$$

If A is an orthogonal transform, then

$$\langle\vec{v}, A\vec{w}\rangle = \langle A^T\vec{v}, \vec{w}\rangle = \langle AA^T\vec{v}, A\vec{w}\rangle$$

so

$$\langle\vec{v} - AA^T\vec{v}, A\vec{w}\rangle = 0 = \langle\vec{0}, A\vec{w}\rangle \text{ for } \underline{\text{all}}\ \vec{w}$$

$\therefore$ if we let $\vec{w}$ run through an orthonormal basis $\vec{e}_1, \dots, \vec{e}_n$

we see $\vec{v} - AA^T\vec{v} = \vec{0}$

so $AA^T\vec{v} = \vec{v} = Id_n\vec{v}$ n×n identity matrix

and $\boxed{AA^T = Id_n}$

this implies $1 = \det(Id_n) = \det(AA^T) = (\det A)^2$

so $\boxed{\det A = \pm 1}$

if $\det A = 1$, we call A a <u>special orthogonal transform</u>

<u>Aside:</u> $O(n) = \{\text{orthogonal transforms of } \mathbb{R}^n\}$
$SO(n) = \{\text{special} \quad \text{"} \quad \text{"}\}$

are examples of <u>Lie groups</u>, the study of these is a beautiful and deep area of math

<u>Isometries of $\mathbb{R}^2$:</u>

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a special orthogonal transformation

then $1 = \det A = ad - bc$

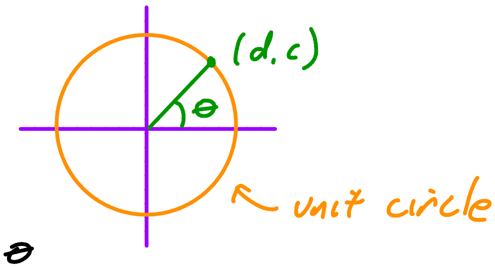
and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$

so we have $a^2+b^2=1$

$c^2+d^2=1$

$ac+bd=0$

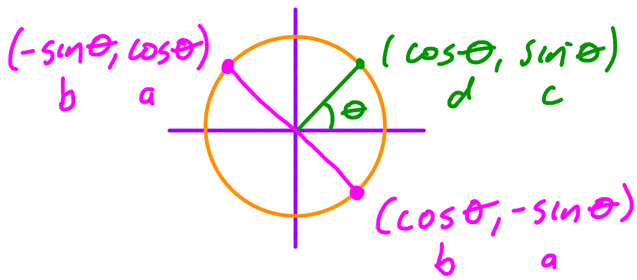
$ad-bc=1$



$\exists$ unique angle θ st. $d=\cos\theta$

$c=\sin\theta$

now $\begin{bmatrix} b \\ a \end{bmatrix} \cdot \begin{bmatrix} d \\ c \end{bmatrix} = 0$ so $\begin{bmatrix} b \\ a \end{bmatrix}$ is a unit vector orthogonal to $\begin{bmatrix} d \\ c \end{bmatrix}$



finally $ad-bc=1 \Rightarrow a=\cos\theta$

$b=-\sin\theta$

so $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

and A corresponds to a rotation about the origin by angle θ

<u>exercise</u>: if A not special, but just orthogonal, then

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{reflect about } x\text{-axis}} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{some } \theta$$

so rigid motions of $\mathbb{R}^2$ are compositions of:

rotations,
translations, and
reflections about x-axis

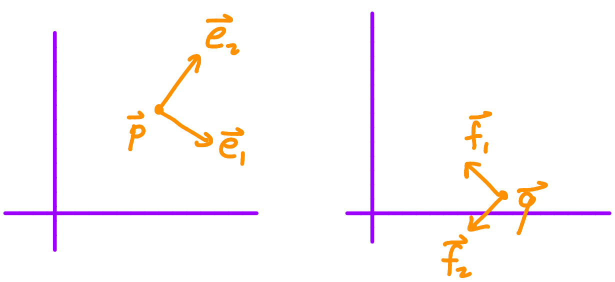
exercise: Isometries of $\mathbb{R}^3$ are compositions of

rotations about some line,
translations,
reflections about xy-plane
reflections through the origin

exercise: let $\vec{e}_1 \dots \vec{e}_n$ be any orthonormal basis for $\mathbb{R}^n$ based at a point $\vec{p} \in \mathbb{R}^n$

and $\vec{f}_1 \dots \vec{f}_n$ be another orthonormal basis for $\mathbb{R}^n$ based at a point $\vec{q} \in \mathbb{R}^n$

Then there is an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that



$$\phi(\vec{p}) = \vec{q}$$

$$\underbrace{D\phi_{\vec{p}}}_{\text{total derivative of } \phi \text{ at } \vec{p}}(\vec{e}_2) = \vec{f}_1$$

Recall: given a function

$$\vec{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

we can write it

$$\vec{F}(x_1, \dots x_n) = (f_1(x_1 \dots x_n), \dots, f_m(x_1, \dots x_n))$$

then

$$D\vec{F}_{\vec{p}} : \underset{\substack{\text{vectors based} \\ \text{at } \vec{p}}}{\mathbb{R}^n} \longrightarrow \underset{\substack{\text{vetors based} \\ \text{at } \vec{q}}}{\mathbb{R}^m}$$

is a linear map that can be expressed as the $m \times n$ matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{p}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{p}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{p}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{p}) \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}(\vec{p}) \right)$$

Hint: consider the case where $\vec{e}_1 \dots \vec{e}_n$ is the standard basis and $\vec{p} = 0$
then consider

$$\phi(\vec{v}) = \vec{q} + A\vec{v}$$

where $A = (\vec{f}_1, \dots, \vec{f}_n)$ expressed in $\vec{e}_1, \dots, \vec{e}_n$