II Curves

A. Curves in $\mathbb{R}^n$

A parameterized curve is a continuous function

$$\vec{\alpha}: I \longrightarrow \mathbb{R}^n$$

where

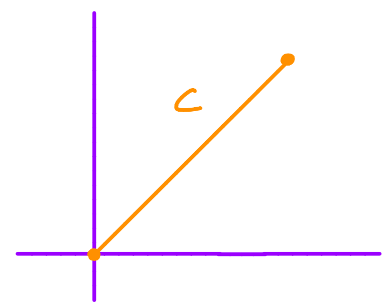
$$I = (a,b) \text{ or } [a,b]$$

the image C of $\vec{\alpha}$ is called a curve

example: $\vec{\alpha}(t) = (t,t) \qquad t \in [0,1]$

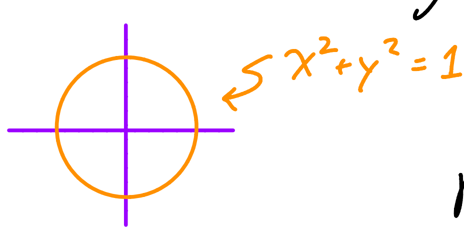
$\vec{\beta}(t) = (\frac{1}{2}t, \frac{1}{2}t) \qquad t \in [0, \frac{1}{2}]$

$\vec{\gamma}(t) = (t^2, t^2) \qquad t \in [0,1]$



note:

1) all 3 functions have the same image, so a given curve can be described by many different functions
2) we say that $\vec{\alpha}$ (or $\vec{\beta}$ or $\vec{\gamma}$ ...) parameterizes the curve C
3) We can think of C as the path of a particle or a piece of wire in $\mathbb{R}^n$
4) we frequently confuse C and $\vec{\alpha}$ but remember we are really interested in C, $\vec{\alpha}$ is just a convenient way to describe C mathematically
5) Curves do not need to be given by a parameterization

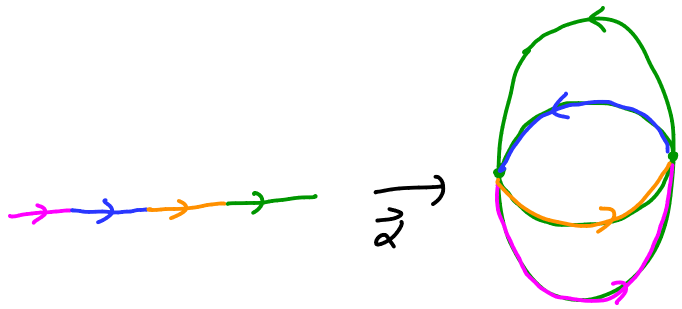
e.g. 

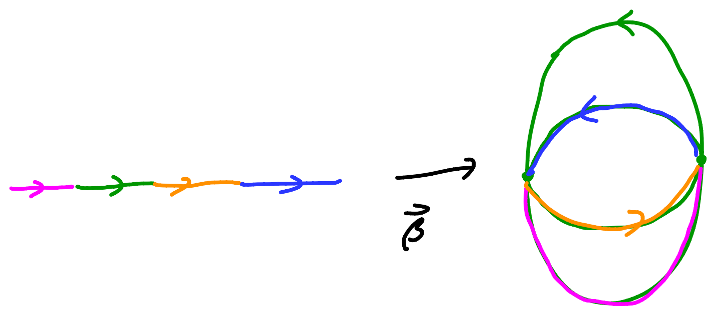
but of course we can parameterize it

$\vec{\alpha}(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

Remark: There is a subtlety in the definition of curve really it is not just the image of $\vec{\alpha}$ but the "trajectory" of the particle traveling along C

example:





the image of $\vec{\alpha}$ and $\vec{\beta}$ are the same but the order in which parts of the path are traversed is different so we will say these are different curves

So really we should think of a curve C as the image of some $\vec{\alpha}:[a,b]\to\mathbb{R}^n$ together with the order in which the points on C are encountered

examples:

1) **Straight lines**

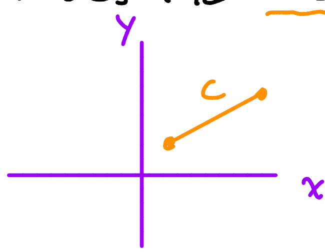
given points $\vec{p}$ and $\vec{q}$ in $\mathbb{R}^n$

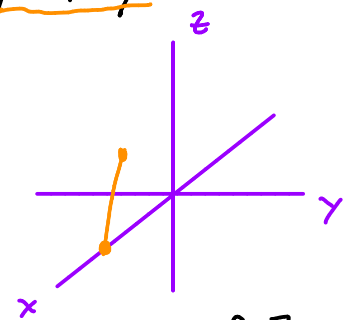
(think of them as vectors)

then $\vec{\alpha}(t) = (1-t)\vec{p} + t\vec{q}$ $\quad t \in [0,1]$

parameterizes the line from $\vec{p}$ to $\vec{q}$

e.g.





$$\vec{\alpha}(t) = (1-t)\begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2t+1\\t+1\end{bmatrix}$$

$$\vec{\alpha}(t) = (1-t)\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}1-t\\-t\\t\end{bmatrix}$$

2) **Circles**

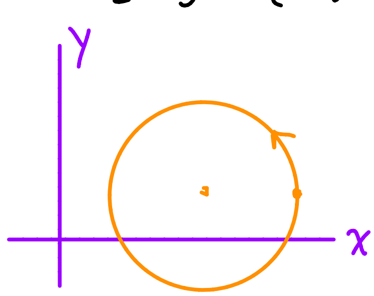
given $r \in \mathbb{R}$, $r > 0$

$\vec{p} \in \mathbb{R}^2$

then $\vec{\alpha}(t) = \vec{p} + (r\cos t, r\sin t)$ $\quad t \in [0, 2\pi]$

parameterizes the circle of radius r about $\vec{p}$

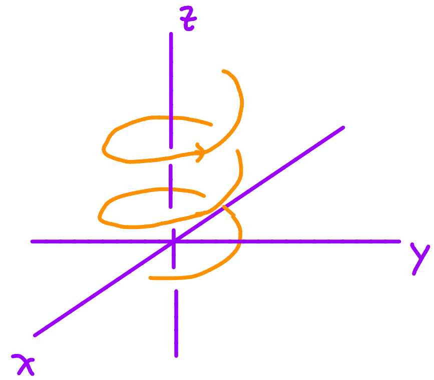
e.g. $\vec{\alpha}(t) = \begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}2\cos t\\2\sin t\end{bmatrix} = \begin{bmatrix}3+2\cos t\\1+2\sin t\end{bmatrix}$



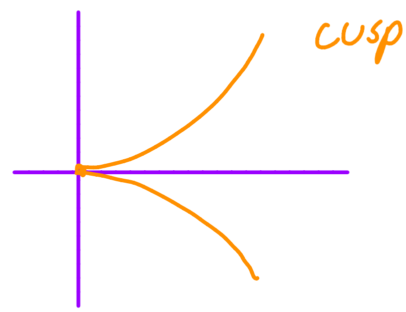
3) Helix

given $r, h \in \mathbb{R}$, $r > 0$, $h \geq 0$

set $\vec{\alpha}(t) = (r\cos t, r\sin t, ht) \qquad t \in \mathbb{R}$



4) $\vec{\alpha}(t) = (t^2, t^3)$
$t \in \mathbb{R}$



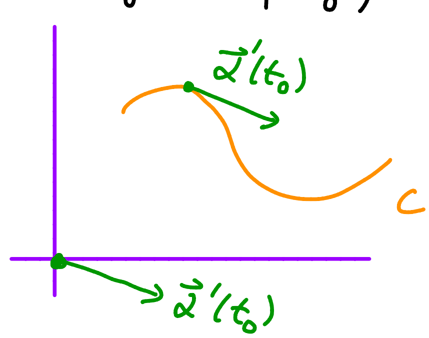
given a curve C, parameterized by a function

$$\vec{\alpha} : I \longrightarrow \mathbb{R}^n$$

$$\vec{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$$

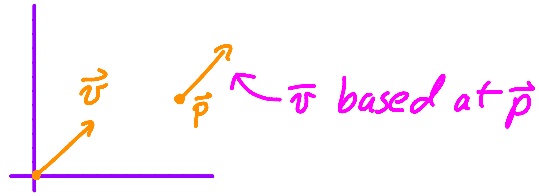
recall from calculus that at the point $\vec{\alpha}(t_0) \in C$ a tangent vector to C is given by

$$\vec{\alpha}'(t_0) = (\alpha_1'(t_0), \ldots, \alpha_n'(t_0))$$



Remarks: 1) Actually $\vec{\alpha}'(t_0)$ is a vector based at $(0, 0, \ldots, 0)$

When we say a vector $\vec{v}$ is based at $\vec{p}$, then we shift $\vec{v}$ so its "tail" is at $\vec{p}$



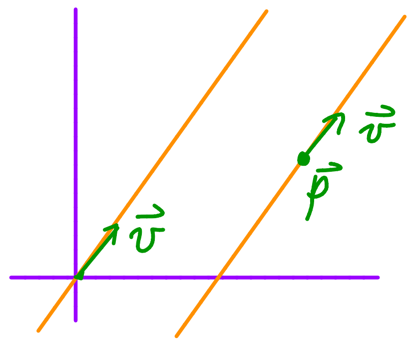
so we should say $\vec{\alpha}'(t_0)$ based at $\vec{\alpha}(t_0)$ is tangent to C

2) If $\vec{v} \neq 0$, then the line spanned by $\vec{v}$ is

$$l_{\vec{v}} = \{ r\vec{v} \mid r \in \mathbb{R} \}$$

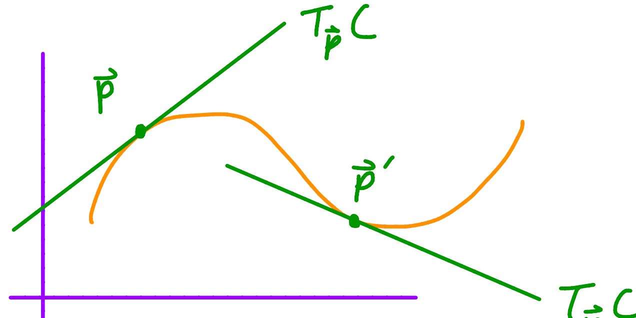
if $\vec{v}$ is based at $\vec{p}$ then the line through $\vec{p}$ in the direction of $\vec{v}$ is

$$\{ r\vec{v} + \vec{p} \mid r \in \mathbb{R} \}$$



so the tangent line to the curve C at $\vec{p} = \vec{\alpha}(t_0)$ is given by

$$T_{\vec{p}}C = \{ \vec{\alpha}(t_0) + r\,\vec{\alpha}'(t_0) \mid r \in \mathbb{R} \}$$



Recall: the tangent line to C at $\vec{p}$ is the "line that best approximates C at $\vec{p}$"

a parameterized curve $\vec{\alpha}(t)$ is called regular if $\vec{\alpha}'(t)$ exists and is non-zero for all t

we need this to talk about tangent lines and many other things

so we usually assume curves are regular (except maybe at a finite number of points)

note: let $\vec{\alpha}: I \to \mathbb{R}^n$ parameterize a curve

if $\vec{\alpha}''(t) = 0$ for all t, then $\vec{\alpha}$ parameterizes (part of) a line

to see this note

$$\vec{\alpha}''(t) = (\vec{\alpha}_1''(t), \dots, \vec{\alpha}_n''(t)) = (0, \dots, 0)$$

so $\quad \alpha_i''(t) = 0 \qquad \alpha_1 : I \to \mathbb{R}$

integrate to get

$$\alpha_i'(t) = \int \alpha_i''(t)\,dt + v_i = v_i \qquad \text{for some constant } v_i$$

so

$$\alpha_i(t) = \int \alpha_i'(t)\,dt + p_i = v_i t + p_i$$

that is

$$\vec{\alpha}(t) = \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} + t \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{p} + \vec{v} t$$

so the second derivative tells us how far $\vec{\alpha}$ is from being a line

Problem: $\vec{\alpha}''$ is not a geometric quantity!

i.e. it depends on the parameterization not just C

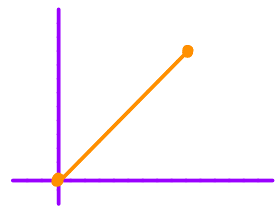
e.g. $\vec{\alpha}(t) = (t,t) \qquad t \in [0,1]$

$\vec{\beta}(t) = (t^2, t^2) \qquad t \in [0,1]$

$\vec{\alpha}''(t) = (0,0) \neq \vec{\beta}''(t) = (2,2)$

also $\|\vec{\alpha}''\| = 0 \qquad \|\vec{\beta}''\| = 2\sqrt{2}$

but both give



so $\vec{\alpha}''$ doesn't necessarily give us information about the curve C it parameterizes (e.g. can't tell example is a line from $\vec{\beta}''$)

to fix this we need to consider arc length

Recall from calculus that if C is parameterized by

$$\vec{\alpha} : [a,b] \to \mathbb{R}^n$$

then the length of C is given by

$$\text{length}(C) = \int_a^b \|\vec{\alpha}'(t)\| \, dt$$

or more generally the length along C from the endpoint $\vec{\alpha}(a)$ to $\vec{\alpha}(s)$ is

$$\ell(s) = \int_a^s \|\vec{\alpha}'(t)\| \, dt$$

lemma 1:

If $\vec{\alpha}$ is a regular parameterization of a curve C, then we can reparameterize C by another function

$$\vec{\beta} : [0, \ell] \to \mathbb{R}^n$$

such that $\|\vec{\beta}'(s)\| = 1$ for all s

Remarks:

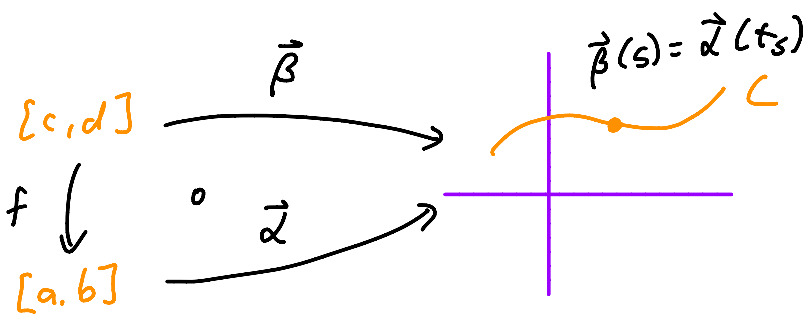
1) if $\vec{\alpha}:[a,b]\to\mathbb{R}^n$ parameterizes a curve C and

$\vec{\beta}:[c,d]\to\mathbb{R}^n$ is another parameterization of C

then we say $\vec{\beta}$ is a reparameterization of C

note: if $\vec{\alpha}$ is one-to-one (i.e. $\vec{\alpha}(t_1)=\vec{\alpha}(t_2)$ then $t_1=t_2$)

then for each $s\in[c,d]$, there is a unique $t_s\in[a,b]$ such that $\vec{\alpha}(t_s)=\vec{\beta}(s)$



so set $f:[c,d]\to[a,b]:s\mapsto t_s$

and we see

$$\boxed{\vec{\beta}(s)=\vec{\alpha}(f(s))=\vec{\alpha}\circ f(s)}$$

Conversely, given any function $h:[k,l]\to[a,b]$

$\vec{\alpha}\circ h:[k,l]\to\mathbb{R}^n$

is a reparameterization of $\vec{\alpha}$

2) We say that $\vec{\alpha}:[a,b]\to\mathbb{R}^n$ parameterizes C by arc length

if $\|\vec{\alpha}'(s)\| = 1$ for all $s \in [0, \ell]$

note: given such an $\vec{\alpha}$ we have

$$\ell(s) = \int_0^s \|\vec{\alpha}'(x)\| dx = s$$

i.e. length of C from $\vec{\alpha}(0)$ to $\vec{\alpha}(s)$ is s

so lemma says regular curves can always be parameterized by arc length

Proof: given $\vec{\alpha} : [a,b] \to \mathbb{R}^n$ parameterizing C

let $f(t) = \int_a^t \|\vec{\alpha}'(x)\| dx$

the fundamental theorem of calculus says

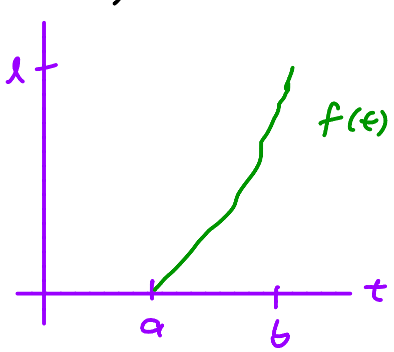
$$\frac{df}{dt} = \|\vec{\alpha}'(t)\| > 0$$

since $\vec{\alpha}$ regular

so f is increasing on $[a,b]$

$\therefore$ f is one-to-one

if $\ell = f(b) =$ length of C then f also onto $[0, \ell]$



exercise: $f(t)$ has an inverse

(recall this is a function

$g : [0, \ell] \to [a,b]$

s.t. $f \circ g(s) = s$ and $g \circ f(t) = t$)

chain rule gives

$$\frac{dg}{ds}(f(t)) \frac{df}{dt}(t) = \frac{d}{dt}(g \circ f)(t) = \frac{d}{dt} t = 1$$

so $\frac{dg}{ds}(s) = \frac{1}{\frac{df}{dt}(t)}$ where $s = f(t)$ and $t = g(s)$

now set $\vec{\beta}(s) = \vec{\alpha}(g(t))$

so $\vec{\beta} : [0,\ell] \to \mathbb{R}^n$ parameterizes C

and

$$\|\vec{\beta}'\| = \|\vec{\alpha}'(g(s))\, g'(s)\| = \left\|\vec{\alpha}'(s)\frac{1}{\frac{df}{dt}(t)}\right\| = \|\vec{\alpha}'(t)\| \frac{1}{\left|\frac{df}{dt}(t)\right|}$$

$$= \|\vec{\alpha}'(t)\| \frac{1}{\|\vec{\alpha}'(t)\|} = 1$$

example: Helix

$\vec{\alpha}(t) = (r\cos t, r\sin t, bt) \qquad t \in [0,\infty)$

$\vec{\alpha}'(t) = (-r\sin t, r\cos t, b)$

$\|\vec{\alpha}'(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t + b^2} = \sqrt{r^2+b^2}$

so $f(t) = \int_0^t \sqrt{r^2+b^2}\, dx = \sqrt{r^2+b^2}\, t$

the inverse of f is

$$f^{-1}(s) = \frac{1}{\sqrt{r^2+b^2}} s$$

so $\vec{\beta}(s) = \vec{\alpha}(f^{-1}(s)) = \left(r\cos\frac{1}{\sqrt{r^2+b^2}}s, r\sin\frac{1}{\sqrt{r^2+b^2}}s, \frac{bs}{\sqrt{r^2+b^2}}\right)$

is a parameterization by arc length

Notation: When we use the variable s in a parameterization of a curve we will always mean we have parameterized by arc length, where t is used for any parameterization

now given a parameterization $\vec{\beta} : [0,\ell] \to \mathbb{R}^n$ of a curve C by arc length we say

$$\vec{T}(s) = \beta'(s)$$

is the unit tangent vector

lemma 2:

$\boxed{\vec{T}'(s) \text{ is perpendicular to } \vec{T}(s)}$

Proof:

$$\vec{T}(s)\cdot\vec{T}(s) = \|\vec{T}(s)\|^2 = 1$$

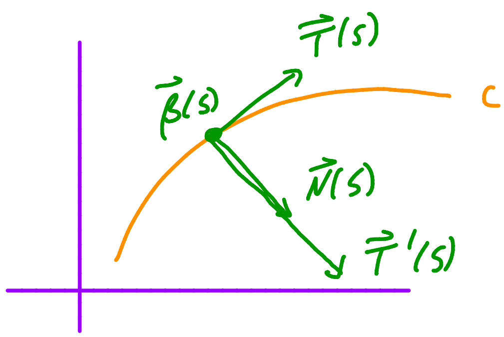
the product rule gives

$$0 = \frac{d}{ds}1 = \frac{d}{ds}\vec{T}\cdot\vec{T} = \left(\frac{d}{ds}\vec{T}\right)\cdot\vec{T} + \vec{T}\cdot\left(\frac{d}{ds}\vec{T}\right)$$

$$= 2\,\vec{T}'(s)\cdot\vec{T}(s)$$

so $\vec{T}'\cdot\vec{T} = 0$ ∎

We call $\vec{N}(s) = \frac{1}{\|\vec{T}'(s)\|}\vec{T}'(s)$ the unit normal vector to C at $\vec{\beta}(s)$



We call $K(s) = \|\vec{T}'(s)\|$ the curvature of C at $\beta(s)$

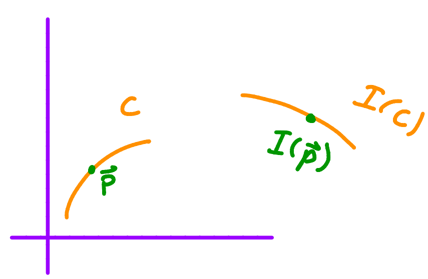
note: $K(s) = \|\vec{\alpha}''(s)\|$ is the acelleration of a particle moving on C with unit speed

so you feel the curvature of a road while driving!

exercise:

this says K is really a geometric quantity telling you how C sits in $\mathbb{R}^n$

1) Show the curvature of a curve C is independent of param.
2) Show if $I:\mathbb{R}^n\to\mathbb{R}^n$ is an isometry then the curvature of C at $\vec{p}$ is the same as the curvature of $I(C)$ at $I(\vec{p})$



3) If $\vec{\alpha}:[a,b]\to\mathbb{R}^n$ is any regular parameterization of C then show you can calculate the curvature of C at $\vec{\alpha}(t)$ by

$$\boxed{\kappa(t) = \left\| \left(\frac{\vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\|}\right)' \frac{1}{\|\vec{\alpha}'(t)\|} \right\|}$$

4) Show directly that the formula in 3) is independent of parameterization

Thm 3:

a regular curve C is (part of) a line
iff
curvature of C is 0 at all points

Remark: So curvature is precisely the measure of how far a curve is from being a line!

Proof: ($\Rightarrow$) If C is a line from $\vec{p}$ to $\vec{q}$ then

$$\vec{\beta}(s) = \frac{\ell - s}{\ell}\vec{p} + \frac{s}{\ell}\vec{q}$$

for $s\in[0,\ell]$ $\quad \ell = \|\vec{p}-\vec{q}\|$

is an arc length parameterization of C

$$\vec{\beta}'(s) = -\frac{1}{\ell}\vec{p} + \frac{1}{\ell}\vec{q}$$

so

$$\kappa(s) = \|\vec{\beta}''(s)\| = \|\vec{0}\| = 0 \checkmark$$

($\Leftarrow$) let $\vec{\beta}(s)$ be an arc length parameterization of C

assume $\kappa(s)=0$ so $\vec{\beta}''(s)=\vec{0}$

then we saw earlier that $\vec{\beta}(s)$ parameterizes part of a line ▦

example: recall an arc length parameterization of the Helix is

$$\vec{\beta}(s) = \left(r\cos\frac{s}{\sqrt{r^2+b^2}},\ r\sin\frac{s}{\sqrt{r^2+b^2}},\ \frac{bs}{\sqrt{r^2+b^2}}\right)$$

so
$$\vec{\beta}'(s) = \left(\frac{-r}{\sqrt{r^2+b^2}}\sin\frac{s}{\sqrt{r^2+b^2}},\ \frac{r}{\sqrt{r^2+b^2}}\cos\frac{s}{\sqrt{r^2+b^2}},\ \frac{b}{\sqrt{r^2+b^2}}\right)$$

and
$$\vec{\beta}''(s) = \left(-\frac{r}{r^2+b^2}\cos\frac{s}{\sqrt{r^2+b^2}},\ \frac{-r}{r^2+b^2}\sin\frac{s}{\sqrt{r^2+b^2}},\ 0\right)$$

thus
$$\kappa(s) = \|\vec{\beta}''(s)\| = \frac{|r|}{r^2+b^2}$$

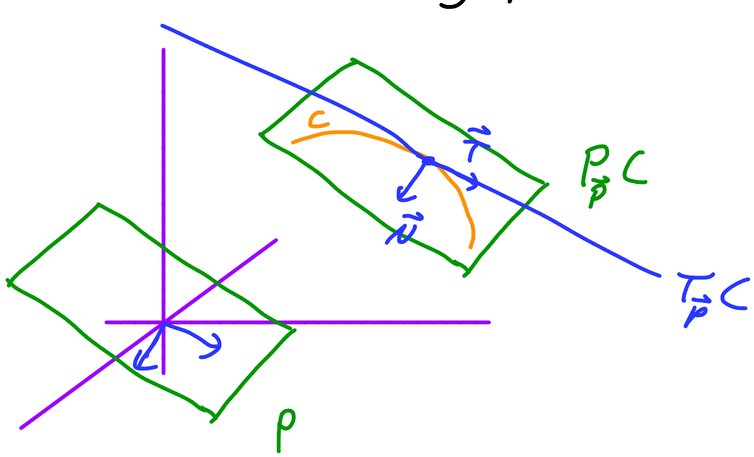
Again let
$$\vec{\beta} : [0, \ell] \longrightarrow \mathbb{R}^n$$
be an arc length parameterization of some curve C

recall $\vec{T}(s) = \vec{\beta}'(s)$ and $\vec{N}(s) = \frac{1}{\kappa(s)}\vec{T}(s)$ are unit orthonormal vectors in $\mathbb{R}^n$

so they span a plane in $\mathbb{R}^n$
$$P = \{a\vec{T}(s_0) + b\vec{N}(s_0) \mid a, b \in \mathbb{R}\}$$

translate P so that it goes through $\vec{\beta}(s_0) = \vec{p}$
$$P_{\vec{p}}C = \{\beta(s_0) + a\vec{T}(s_0) + b\vec{N}(s_0) \mid a, b \in \mathbb{R}\}$$

this is called the <u>osculating plane</u> to C at $\vec{p} = \vec{\beta}(s_0)$



<u>note</u>: $P_{\vec{p}}C$ contains the tangent line $T_{\vec{p}}C$

<u>exercise</u>: Convince yourself that $P_{\vec{p}}C$ is the plane that C

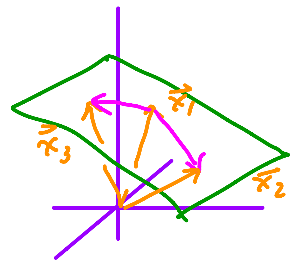
comes closest to lying in at $\vec{p}$

(later we will see precisely when C lies in $P_{\vec{p}}C$)

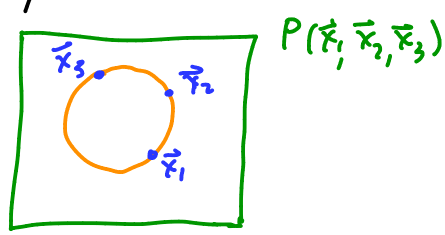
Recall: given 3 points $\vec{x}_1, \vec{x}_2, \vec{x}_3$ in $\mathbb{R}^n$ that do not lie on a line then they

① determine a unique plane

$$P(\vec{x}_1,\vec{x}_2,\vec{x}_3) = \vec{x}_1 + \text{span}\{\vec{x}_2-\vec{x}_1, \vec{x}_3-\vec{x}_1\}$$



② determin a unique circle $C(\vec{x}_1,\vec{x}_2,\vec{x}_3)$ in $P(\vec{x}_1,\vec{x}_2,\vec{x}_3)$



Facts: let $\vec{\beta}:[0,\ell]\to\mathbb{R}^n$ be a regular parameterization of C

suppose $s_0\in[0,\ell]$ s.t. $\mathcal{X}(s_0)\neq 0$

I) for points $s_1, s_2, s_3 \in [0,\ell]$ sufficiently close to s_0

$\vec{\beta}(s_1), \vec{\beta}(s_2), \vec{\beta}(s_3)$ donot lie on a line

II) the osculating plane is

$$P_{\vec{\beta}(s_0)}C = \lim_{s_1,s_2,s_3\to 0} P(\vec{\beta}(s_1),\vec{\beta}(s_2),\vec{\beta}(s_3))$$

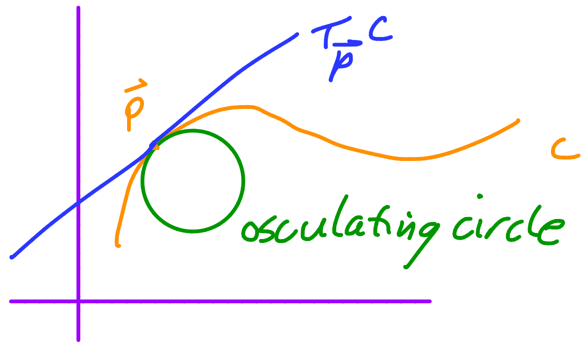
III) The limit $\displaystyle\lim_{s_1,s_2,s_3\to s_0} C(\vec{\beta}(s_1),\vec{\beta}(s_2),\vec{\beta}(s_3))$

is a circle $C_{\vec{\beta}(s_0)}$ in $P_{\vec{\beta}(s_0)}C$

it is called the osculating circle and can be parameterized by

$$\vec{\alpha}(t) = \vec{\beta}(s_0) + \frac{1}{\mathcal{X}(s_0)}\vec{N}(s_0) + \frac{1}{\mathcal{X}(s_0)}\left[(\sin t)\,\vec{T}(s_0) + (\cos t)\,\vec{N}(s_0)\right]$$

so the circle has radius $\frac{1}{\kappa(s_0)}$



<u>note</u>: 1) osculating circle is tangent to C at $\vec{p}$
(has "order 2" contact with C)

2) if κ is close to 0 then C is almost straight
if κ is large then C curves a lot.