B. Curves in $\mathbb{R}^2$

1. Local Theory: signed curvature

for plane curves we can refine our notion of curvature

an orientation on a curve C is a direction

note a parameterization $\vec{\alpha}$ of C gives an orientation on C

exercise: an oriented curve has a unique arc lenght parameterization inducing the given orientation (once starting pt fixed)

let C be an oriented curve and $\vec{\alpha}:[0,\ell]\to\mathbb{R}^2$ an arc length parameterizaton inducing the orientation

so $\vec{T}(s)=\vec{\alpha}'(s)$ is the unit tangent vector to C at $\vec{\alpha}(s)$

set $\hat{N}(s)$ = vetor $\vec{T}(s)$ rotated $90°$ counterclockwise

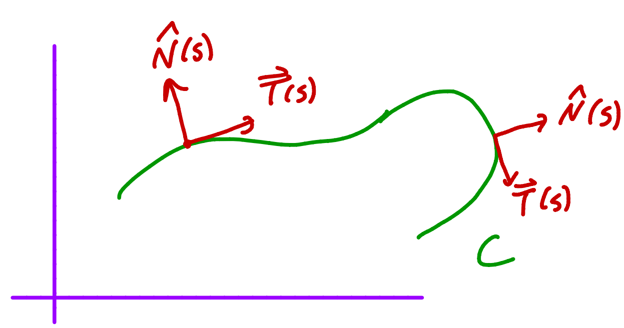
use $\hat{N}(s)$ since $\vec{N}=\frac{\vec{T}'}{\|\vec{T}'\|}$ earlier

(we denote "rotation by $90°$ counterclockwise" by

$$i:\mathbb{R}^2\to\mathbb{R}^2 \quad \text{"complex multiplication"}$$

so $\hat{N}(s)=i\,\vec{T}(s)$

if $\vec{v}=\begin{bmatrix} v_1\\ v_2\end{bmatrix}$ then $i\vec{v}=\begin{bmatrix} -v_2\\ v_1\end{bmatrix}$)



Recall lemma 2 says $\vec{T}'(s)$ is perpendicular to $\vec{T}(s)$

in $\mathbb{R}^2$ this means that $\vec{T}'(s)$ and $\hat{N}(s)$ are parallel

that is there is some number $\kappa_\sigma(s)$ such that

$$\vec{\alpha}''(s) = \vec{T}'(s) = \kappa_\sigma(s)\,\hat{N}(s)$$

note: $\vec{N}(s) = \dfrac{\vec{T}'(s)}{\|\vec{T}'(s)\|}$ is $\pm \hat{N}(s)$

(only 2 unit normal vectors to a line in $\mathbb{R}^2$)

so $\kappa(s) = |\kappa_\sigma(s)|$ (recall $\kappa(s) = \|\vec{T}'(s)\| \geq 0$)

so we call $\kappa_\sigma(s)$ the signed curvature of the curve C (or $\vec{\alpha}$)

Remark: if $\kappa_\sigma(s) > 0$, then C is "turning towards $\hat{N}(s)$"

if $\kappa_\sigma(s) < 0$, then C is "turning away from $\hat{N}(s)$"

Theorem 4:

an oriented curve C is part of a circle (line)

$\Longleftrightarrow$

κ_σ is a non-zero (zero) constant

moreover, if κ_σ is a non-zero constant, then

C is part of a circle of radius $\dfrac{1}{|\kappa_\sigma|}$

and if $\kappa_\sigma > 0$, C is oriented counterclockwise

if $\kappa_\sigma < 0$, C is oriented clockwise

exercise:

1) Show that a circle of radius R centered at $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$

is parameterized by arc length by

$$\vec{\alpha}(s) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} R\cos\frac{1}{R}s \\ R\sin\frac{1}{R}s \end{bmatrix} \qquad s \in [0, 2\pi R]$$

2) this parameterization is counterclockwise

find the clockwise parameterization $\vec{\beta}(s)$

3) show $K_\sigma(s) = \frac{1}{R}$ for $\vec{\alpha}$ and $K_\sigma(s) = -\frac{1}{R}$ for $\vec{\beta}$

note: This completes ($\Rightarrow$) in the theorem

for ($\Leftarrow$) we will use

Th^{m} 5 (Fundamental theorem of plane curves):

given: 1) $I = [0, \ell] \subset \mathbb{R}$

2) $c: I \to \mathbb{R}$ a continuous function

3) $\vec{p}, \vec{v} \in \mathbb{R}^2$ with $\|\vec{v}\| = 1$

Then there exists a unique curve C with a arc length parameterization $\vec{\alpha}: I \longrightarrow \mathbb{R}^2$

such that 1) $\vec{\alpha}(0) = \vec{p}$

2) $\vec{\alpha}'(0) = \vec{v}$

3) $K_\sigma(s) = c(s)$

Corollary 6:

If C_1 and C_2 are two regular oriented plane curves of length L and they have the same signed curvature

then there is some isometry ("rigid motion")

$$\phi: \mathbb{R}^2 \to \mathbb{R}^2$$

such that

$$\phi(C_1) = C_2$$

Remark: 1) So signed curvature determines a plane curve upto rigid motion!

2) if two curves of the same lenght have the same signed curvature and are tangent at their starting point, then they are the same curve!

Proof of Corollary:

let $\vec{\alpha}:[0,L]\to\mathbb{R}^2$ be an arc length parameterization of C_1

$\vec{\beta}:[0,L]\to\mathbb{R}^2$ " " " " C_2

let $\vec{p}_1=\vec{\alpha}(0)$, $\vec{v}_1=\vec{\alpha}'(0)$

$\vec{p}_2=\vec{\beta}(0)$, $\vec{v}_2=\vec{\beta}'(0)$

earlier we saw that there was an isometry

$\phi:\mathbb{R}^2\to\mathbb{R}^2$ such that $\phi(\vec{p}_1)=\vec{p}_2$

$D\phi_{\vec{p}_1}(\vec{v}_1)=\vec{v}_2$

recall $\phi(\vec{p})=A\vec{p}+\vec{a}$

show A can be taken to be a special orthogonal transform

set $\vec{\gamma}(s)=\phi\circ\vec{\alpha}(s)$

note: $\vec{\gamma}(0)=\phi(\vec{p}_1)=\vec{p}_2$

$\vec{\gamma}'(0)=D\phi_{\vec{\alpha}(0)}(\vec{\alpha}'(0))=D\phi_{\vec{p}_1}(\vec{v}_1)=\vec{v}_2$

signed curvature of $C_3=$ image $\vec{\gamma}$ at $\vec{\gamma}(s)$ is the same as signed curvature of C_1 at $\vec{\alpha}(s)$

exercise: Prove this if it is not clear to you

(i.e. show that signed curvature does not change under isometry with A special orthog.)

Hint: $\vec{\gamma}(s)=A\cdot\vec{\alpha}(s)+\vec{a}$ A special orthogonal transform and point $\vec{a}$

now compute $K_o(s)$

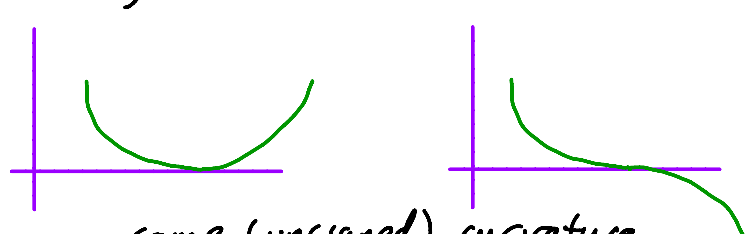
If A not special (i.e. $\det A=-1$) show K_o changes sign

so $\vec{\beta}$ and $\vec{\gamma}$ start at same point with same tangent vector and same signed curvature so uniqueness in Th$^{\underline{m}}$5 gives $\vec{\gamma}(s)=\vec{\beta}(s)$ so $C_2=C_3=\phi(C_1)$ ▦

exercise: Complete the proof of Th$^{\underline{m}}$4 using Th$^{\underline{m}}$5 (or Cor 6)

Remark: In Th$^{\underline{m}}$5 (unsigned) curvature is not enough to determine a curve

e.g.



same (unsigned) curvature

Proof of Thm 5:

Suppose we are given an arc length parameterization

$$\vec{\beta} : [0, \ell] \longrightarrow \mathbb{R}^2$$

then $\|\vec{\beta}\,'(s)\| = 1$ so for each $s \in [0, \ell]$ there is a $\theta(s)$

such that

$$\vec{\beta}\,'(s) = (\cos\theta(s), \sin\theta(s))$$

so

$$\boxed{\vec{\beta}(s) = \left(a + \int_0^s \cos\theta(t)\,dt,\ b + \int_0^s \sin\theta(t)\,dt\right)} \qquad 1$$

where $\vec{\beta}(0) = (a, b)$

thus $\underline{\theta(s) \text{ and } (a,b) \text{ completely determine } \vec{\beta}(s)}$

note $\vec{\beta}\,''(s) = (-\theta'(s)\sin\theta(s), \theta'(s)\cos\theta(s))$

and $\hat{N}(s) = i\,\vec{T}(s) = i\,\vec{\beta}\,'(s) = (-\sin\theta(s), \cos\theta(s))$

the definition of $\kappa_\sigma(s)$ is

$$\vec{\beta}\,''(s) = \kappa_\sigma(s)\,\hat{N}(s)$$

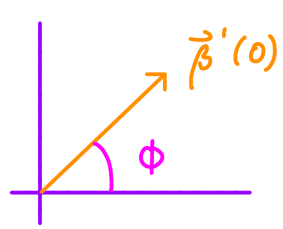
so we see

$$\boxed{\kappa_\sigma(s) = \theta'(s)}$$

thus

$$\boxed{\theta(s) = \phi + \int_0^s \kappa_\sigma(t)\,dt} \qquad 2$$

where



note: $\vec{\beta}\,'(0)$ is a unit vector so is determined by an angle ϕ

so $\vec{\beta}(s)$ is determined by $\kappa_\sigma(s)$, $\vec{\beta}(0)$, and $\vec{\beta}\,'(0)$ by 1 and 2

this proves the uniqueness statement in the theorem

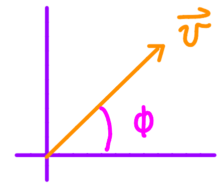
but now existance easy too!

given $c:[0,\ell]\to\mathbb{R}$ continuous

$\vec{p},\vec{v}\in\mathbb{R}^2$ with $\|\vec{v}\|=1$

let ϕ be the angle $\vec{v}$ forms w/ x-axis

and $\vec{p}=(a,b)$



now set

$$\theta(s)=\int_0^s c(t)\,dt+\phi$$

and

$$\vec{\alpha}(s)=\left(a+\int_0^s\cos\theta(t)\,dt,\ b+\int_0^s\sin\theta(t)\,dt\right)$$

we clearly have

$$\vec{\alpha}(0)=(a,b)$$
$$\vec{\alpha}'(0)=(\cos\phi,\sin\phi)=\vec{v}$$

and

$$\vec{\alpha}''(s)=(-c'(s)\sin c(s),\ c'(s)\cos c(s))=c(s)(-\sin c(s),\cos c(s))$$
$$=c(s)(i\,\vec{\alpha}'(s))=c(s)\,\hat{N}(s)$$

so $\chi_\sigma(s)=c(s)$

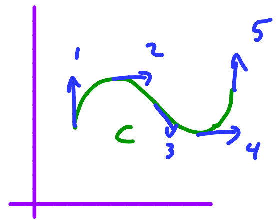
2. Rotation number, total curvature, and regular homotopy

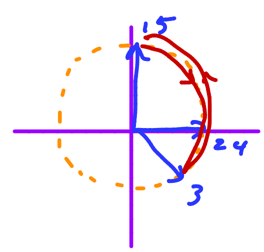
If $\vec{\alpha}:[0,\ell]\to\mathbb{R}^2$ is a unit speed parameterization of a curve C

then $\vec{T}(s)=\vec{\alpha}'(s)$ is also a curve

$$\vec{T}:[0,\ell]\to\mathbb{R}^2$$

the image of $\vec{T}$ is on the unit circle and the curve is called the tantrix of C





just like before, since $\vec{T}(s)$ is a unit vector there is an angle $\theta(s)$

s.t. $\vec{T}(s)=(\cos\theta(s),\sin\theta(s))$

note: $\vec{T}'(s) = (-\theta'(s)\sin\theta(s), \theta'(s)\cos\theta(s))$

$\hat{N}(s) = i\vec{T}(s) = (-\sin\theta(s), \cos\theta(s))$

so the signed curvature is

$$\underline{\kappa_\sigma(s) = \vec{T}'(s)\cdot(i\vec{T}(s)) = \theta'(s)}$$

since $\theta(s)$ is an angle it is only well-defined modulo 2π

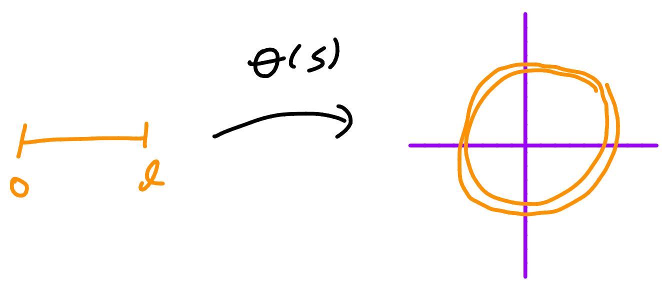
but $\theta'(s)$ is a well-defined number and so we can define the number

$$\theta(s) = \theta(0) + \int_0^s \theta'(t)\,dt$$

some number in $[0, 2\pi)$ corresponding to initial angle

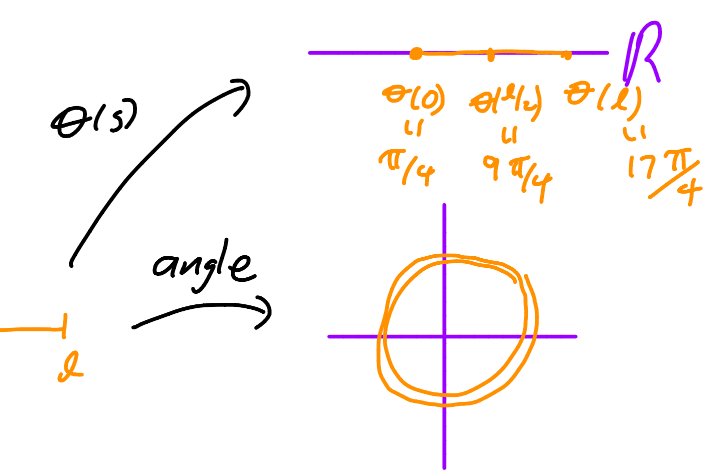
this gives a number for all s and $\theta(s)$ mod 2π represents the angle

example:



"angle" always between 0 and 2π

but can define $\theta(s) \in \mathbb{R}$



we define the rotation number (or index) of a curve C with its arc length parameterization

$$\vec{\alpha} : [0, \ell] \to \mathbb{R}^2$$

by

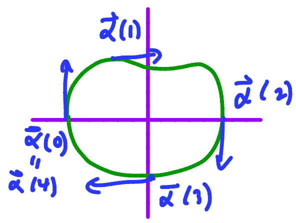
$$R(C) = \frac{1}{2\pi}\left(\theta(\ell) - \theta(0)\right)$$

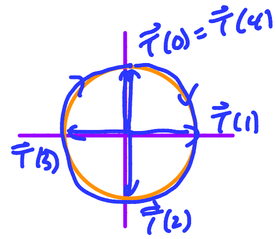
where

$\theta : [0, \ell] \to \mathbb{R}$ is such that

$$\vec{\alpha}\,'(s) = (\cos\theta(s), \sin\theta(s))$$

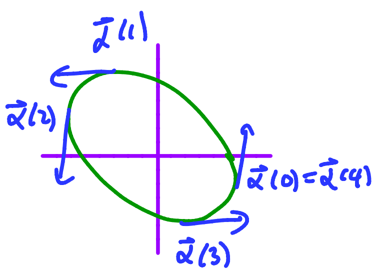
examples:

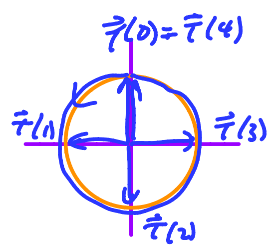




so $\theta(0)$ starts at $\pi/2$ and decreases to $-\frac{3\pi}{2}$

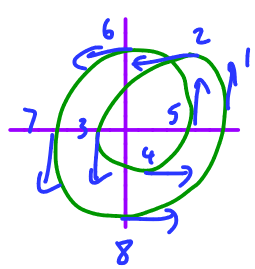
so $R(C) = \frac{1}{2\pi}\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) = -1$

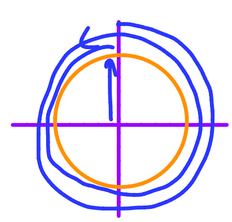




so $\theta(0)$ starts at $\pi/2$ and increases to $\frac{5\pi}{2}$

so $R(C) = \frac{1}{2\pi}\left(\frac{5\pi}{2} - \frac{\pi}{2}\right) = 1$





So $\theta(0)$ starts at $\frac{\pi}{2}$ and increases to $\frac{9\pi}{2}$

so $R(C) = \frac{1}{2\pi}\left(\frac{9\pi}{2} - \frac{\pi}{2}\right) = 2$

we now define the total signed curvature of C to be

$$TK(C) = \int_0^\ell \kappa_\sigma(s)\,ds$$

where C is a regular curve of length ℓ

lemma 7:

> for a regular curve C we have
>
> $$\underbrace{TK(C)}_{\text{total signed curvature}} = 2\pi \underbrace{R(C)}_{\text{rotation number}}$$

Proof: given an arc length parameterization

$$\vec{\alpha} : [0,\ell] \to \mathbb{R}^2$$

of C

we have a function $\theta : [0,\ell] \to \mathbb{R}$ such that

$$\vec{\alpha}'(s) = (\cos\theta(s), \sin\theta(s))$$

as above we know

$$\theta'(s) = \kappa_g(s)$$

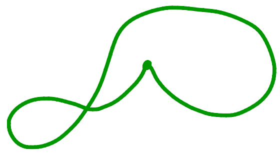
so

$$\begin{aligned} R(C) &= \frac{1}{2\pi}(\theta(\ell) - \theta(0)) \\ &= \frac{1}{2\pi}\int_0^\ell \theta'(s)\,ds \\ &= \frac{1}{2\pi}\int_0^\ell \kappa_g(s)\,ds = \frac{1}{2\pi}\,TK(C) \end{aligned}$$

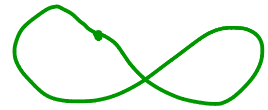
a curve C is called a closed curve if it can be parameterized by a function $\vec{\alpha} : [0,\ell] \to \mathbb{R}^2$ such that $\vec{\alpha}(0) = \vec{\alpha}(\ell)$

it is a regular closed curve if $\vec{\alpha}'(t) \neq 0$ for $t \in [0, \ell]$

and $\vec{\alpha}'(0) = \vec{\alpha}'(\ell)$



closed curve
but not regular



regular closed
curve

lemma 8:

> If C is a regular closed curve, then
> $R(C)$ is an integer
> (and $TK(C)$ is an integral multiple of 2π)

Remark: Surprising! By asking for a curve to close up you are restricting its curvature!

Proof: if $\vec{\alpha} : [0,\ell] \to \mathbb{R}^2$ is an arc length parameterization of C

and $\theta : [0,\ell] \to \mathbb{R}$ a function s.t. $\vec{\alpha}'(s) = (\cos\theta(s), \sin\theta(s))$

then $\vec{\alpha}'(0) = \vec{\alpha}'(\ell)$ means $\theta(0) = \theta(\ell) \mod 2\pi$

ie $\theta(\ell)$ and $\theta(0)$ differ by a integral multiple of 2π #

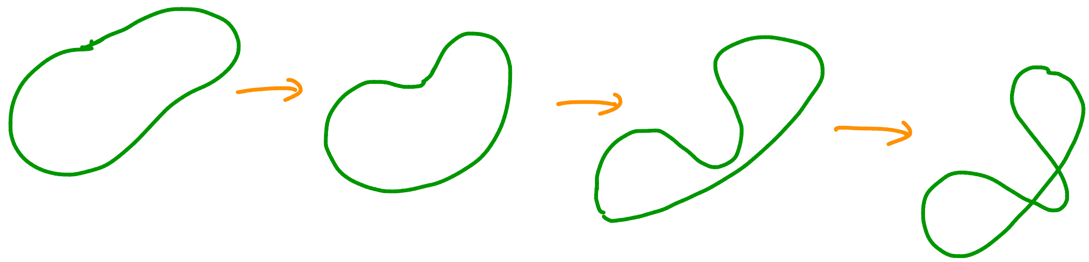
Normally we think of curves as being "equivalent" if there is a rigid motion (isometry) of $\mathbb{R}^2$ taking one to the other

intuition: C is an unbendable wire so you can move it around but that is all

there is another way to think of closed curves as being equivalent

intuition: C is a flexible and stretchy wire so it can be bent and pulled into other shapes

example:



definition: let C_0 and C_1 be regular closed curves given by regular parameterizations

$$\vec{\alpha}_0 : [0,1] \to \mathbb{R}^2$$
$$\vec{\alpha}_1 : [0,1] \to \mathbb{R}^2$$

we say C_0 and C_1 are <u>regularly homotopic</u> if there is a continuous function

$$H(t,x) : [0,1] \times [0,1] \to \mathbb{R}^2$$

such that

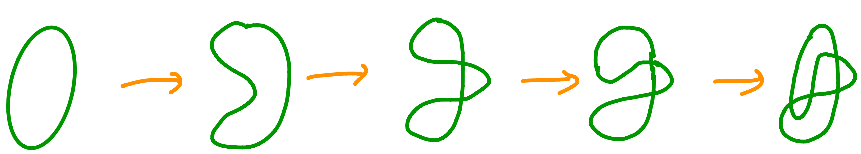
1) $H(t,0) = \vec{\alpha}_0(t)$
2) $H(t,1) = \vec{\alpha}_1(t)$
3) for each fixed x_0 the function

$$h_{x_0} : [0,1] \to \mathbb{R}^2$$
$$t \mapsto H(t, x_0)$$

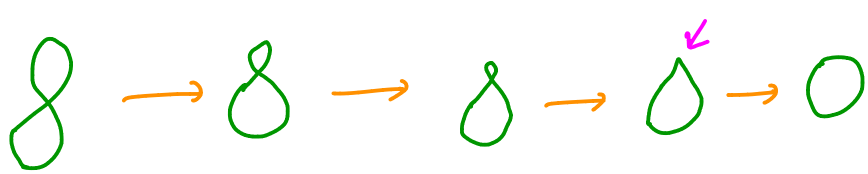
is a regular closed curve

intuition: You can continuously deform one curve into another through regular curves

examples:



regular homotopy



not regular

Lemma 9:

> If C_0 and C_1 are regular homotopic closed curves then $R(C_0) = R(C_1)$

Proof: Using the function $H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2$ from the definition of regular homotopy we get

$$\begin{aligned} R : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \underbrace{R(h_x)}_{\text{rotation number of closed curve } h_x} \end{aligned}$$

exercise: R is continuous

recall $R(x) = R(h_x)$ is an integer for each x by lemma 8

if $R(C_0) = n \neq m = R(C_1)$ the let r be a non-integer between n and m

by the intermediate value theorem for continuous functions there is some $x_0 \in [0,1]$ such that $R(x_0) = r$

this contradicts $R(x_0)$ an integer

$\therefore$ must have $n = m$ $\blacksquare$

Thm 10 (Whitney-Graustein Thm):

> two regular closed curves C_0, C_1 are regularly homotopic
> $$\Leftrightarrow$$
> $$R(C_0) = R(C_1)$$

Remark: 1) "Can't turn a circle inside out"

2) Amazing theorem, says regular homotopy type completely determined by a number!

3) by lemma 7 this says

two regular closed curves are reglar homotopic
$\Longleftrightarrow$
they have the same total signed curvature

K_σ positive and negative

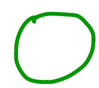
K_σ always positive

but integrals same!

Cor 11:

$$\boxed{\begin{array}{c}\text{a regular closed curve } C \text{ is regularly} \\ \text{homotopic to a simple curve} \\ \Longleftrightarrow \\ TK(C) = \pm 2\pi\end{array}}$$

means embedded

Proof: ($\Rightarrow$) simple curve is regular homotopic to  C'

exercise: "prove this!"

Hint: Hard look it up

so $R(C') = \pm 1$ ✓

($\Leftarrow$) if $TK(C) = \pm 2\pi$ then last $th^{\underline{m}}$ gives C regular homotopic to C' ▦

Proof of $Th^{\underline{m}}$ 10:

($\Rightarrow$) this is exactly lemma 9 ✓

($\Leftarrow$) given C_0 and C_1 with $R(C_0) = R(C_1)$

let $\vec{\alpha}_0$ and $\vec{\alpha}_1$ be arc length parameterizations of C_0 and C_1

below we will show that after regular homotopy we can assume

(A) $\vec{\alpha}_0(0) = \vec{\alpha}_1(0)$ and $\vec{\alpha}_0'(0) = \vec{\alpha}_1'(0)$

(B) length $\vec{\alpha}_0$ = length $\vec{\alpha}_1$

(C) sign of signed curvatures of C_0, C_1 same at 0

in particular not zero

given this let ℓ = length C_0 = length C_1

and $\vec{\alpha}_0(0) = (a,b)$

recall there exist functions

$$\theta_i : [0,\ell] \to \mathbb{R} \qquad i=0,1$$

such that

$$\vec{\alpha}_i'(s) = (\cos\theta_i(s), \sin\theta_i(s)) \qquad i=0,1$$

and since $\vec{\alpha}_0'(0) = \vec{\alpha}_1'(0)$ we can assume $\theta_0(0) = \theta_1(0)$

and $\theta_0(\ell) - \theta_0(0) = 2\pi R(C_0) = 2\pi R(C_1) = \theta_1(\ell) - \theta_1(0)$

$\therefore \; \theta_1(\ell) = \theta_0(\ell)$

let $\Theta : [0,\ell]\times[0,1] \to \mathbb{R}$

$$(s,x) \longmapsto x\,\theta_1(s) + (1-x)\theta_0(s)$$

and $\vec{H} : [0,\ell]\times[0,1] \longrightarrow \mathbb{R}^2$ be defined by

$$\vec{H}(s,x) = \underbrace{(a,b)}_{\text{start at right point}} + \underbrace{\left(\int_0^s \cos\Theta(t,x)\,dt, \int_0^s \sin\Theta(t,x)\,dt\right)}_{\text{main def}^{n}\text{ of curve}}$$

$$\underbrace{- \frac{s}{\ell}\left(\int_0^\ell \cos\Theta(t,x)\,dt, \int_0^\ell \sin\Theta(t,x)\,dt\right)}_{\text{make curve close up}}$$

note: 1) $\vec{\alpha}_0(s) = \left(a + \int_0^s \cos\theta_0(t)\,dt,\; b + \int_0^s \sin\theta_0(t)\,dt\right)$

$\vec{\alpha}_0(\ell) = \vec{\alpha}_0(0) = (a,b)$

so $\int_0^\ell \cos\theta_0(t)\,dt = 0 = \int_0^\ell \sin\theta_0(t)\,dt$

and similarly for $\theta_1(t)$

2) $\vec{H}(s,0) = \vec{\alpha}_0(s)$ and $\vec{H}(s,1) = \vec{\alpha}_1(s)$

3) for a fixed x_0 in $(0,1)$, $\vec{h}_{x_0}(s) = \vec{H}(s, x_0) : [0,\ell] \to \mathbb{R}^2$

is a curve and

$$\left.\begin{aligned} \vec{h}_{x_0}(0) &= (a,b) \\ \vec{h}_{x_0}(\ell) &= (a,b) \end{aligned}\right\} \text{closed curve}$$

we now check it is regular

$$\vec{h}_{x_0}'(t) = \underbrace{\left(\cos\theta(t,x_0), \sin\theta(t,x_0)\right)}_{\vec{A}} - \underbrace{\frac{1}{\ell}\left(\int_0^\ell \cos\theta(t,x_0)\,dt, \int_0^\ell \sin\theta(t,x_0)\,dt\right)}_{\vec{B}}$$

$$\|\vec{A}\| = \sqrt{\cos^2\theta(t,x_0) + \sin^2\theta(t,x_0)} = \sqrt{1} = 1$$

$$\|\vec{B}\| = \left\|\frac{1}{\ell}\int_0^\ell \left(\cos\theta(t,x_0), \sin\theta(t,x_0)\right) dt\right\|$$

$$\overset{*}{\leq} \frac{1}{\ell}\int_0^\ell \left\|\left(\cos\theta(t,x_0), \sin\theta(t,x_0)\right)\right\| dt$$

standard fact from calculus

$$= \frac{1}{\ell}\int_0^\ell 1\,dt = 1$$

get $=$ in $*$ $\Leftrightarrow$ $(\cos\theta(t,x_0), \sin\theta(t,x_0))$ is a multiple of a fixed vector

$\Leftrightarrow$ $(\cos\theta(t,x_0), \sin\theta(t,x_0))$ is constant in t

$\Leftrightarrow$ $\theta(t,x_0)$ constant in t

but $\theta(t,x_0) = x_0\,\theta_1(t) + (1-x_0)\,\theta_0(t)$

so if $\theta(t,x_0)$ constant in t, then

$$\underbrace{\theta_1'(t)}_{\text{curvature of } C_1} = \underbrace{\frac{x_0 - 1}{x_0}}_{<0} \underbrace{\theta_0'(t)}_{\text{curvature of } C_0}$$

since signed curvatures same near 0 this can't be true

so $\|\vec{h}_{x_0}'(s)\| = \|\vec{A} - \vec{B}\| \geq \|\vec{A}\| - \|\vec{B}\| > 0$

and $\vec{h}_{x_0}(s)$ a regular parameterization of a curve

finally $\Theta(\ell, x_0) - \Theta(0, x_0) = x_0 \theta_1(\ell) + (1-x_0)\theta_0(\ell)$
$$- x_0 \theta_1(0) - (1-x_0)\theta_0(\ell)$$
$$= x_0 2\pi R(C_1) + (1-x_0) 2\pi \underbrace{R(C_0)}_{R(C_1)}$$
$$= 2\pi R(C_1)$$

so $\cos \Theta(\ell, x_0) = \cos \Theta(0, x_0)$
$\sin \Theta(\ell, x_0) = \sin \Theta(0, x_0)$

and hence
$$\vec{h}_{x_0}'(\ell) = (\cos \Theta(\ell, x_0), \sin \Theta(\ell, x_0)) - \tfrac{1}{\ell}\left(\int_0^\ell -, \int_0^\ell -\right)$$
$$= (\cos \Theta(0, x_0), \sin \Theta(0, x_0)) - \tfrac{1}{\ell}\left(\int_0^\ell -, \int_0^\ell -\right)$$
$$= \vec{h}_{x_0}'(0)$$

so $\vec{h}_{x_0}$ is a regulor closed curve and $\vec{H}$ is a regular homotopy from $\vec{\alpha}_0$ to $\vec{\alpha}_1$

we are left to check Ⓐ, Ⓑ, and Ⓒ

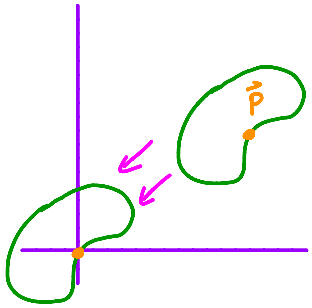
Ⓑ if $\vec{\alpha}_0$ has length ℓ then
$$\vec{\beta}_x(t) = \frac{1}{x\ell + (1-x)} \vec{\alpha}_0(t)$$
is a regular homotopy of $\vec{\alpha}_0$ to $\vec{\beta}_1$ with length 1
(now, reparameterize by arc length)

do same for $\vec{\alpha}_1$ to get same length

Ⓐ and Ⓒ since signed curvature not 0 (since not a line) and total signed curvature same there must be a point $\vec{p}$ on C_0 and $\vec{q}$ on C_1 with same sign of signed curvature now choose parameterizations s.t. $\vec{\alpha}_0(0) = \vec{p}$ and $\vec{\alpha}_1(0) = \vec{q}$

<u>exercise:</u> prove you can do this

so $\vec{\alpha}_0$ and $\vec{\alpha}_1$ have same sign of curvature at 0

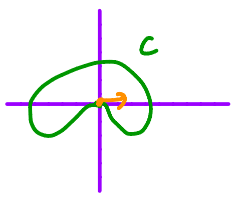


given any point $\vec{p}$ on C_0

$$\beta_x(s) = \vec{\alpha}_0(s) - x\vec{p}$$

is a regular homotopy from $\vec{\alpha}_0$ to $\vec{\beta}_1$ that parameterizes a curve through $\vec{0}$

(since translation doesn't change curvature we have curvature of $\vec{\alpha}_0$ and $\vec{\beta}_1$ same)



similarly we can rotate curve so that

$$\vec{\alpha}_0'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(also does not change curvature)

so after regular homotopy can assume

$$\vec{\alpha}_0(0) = \vec{\alpha}_1(0) = (0,0)$$

$$\vec{\alpha}_0'(0) = \vec{\alpha}_1'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and signed curvatures have same sign near 0