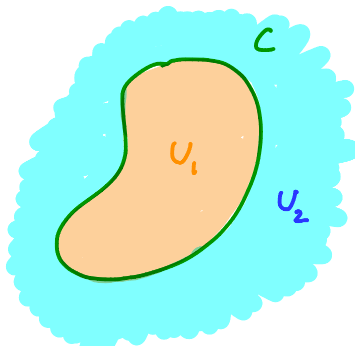
3. Convexity and Curvature

we will begin with an "obvious" statement that takes some time to prove (and we will not prove it)

Th$^{\underline{m}}$ (Jordan curve theorem):

let C be a simple closed plane curve
Then $\mathbb{R}^2 - C$ has two components U_1 and U_2
one, say U_1, is bounded and the other is not

example:



we say U_1 is the region of $\mathbb{R}^2$ bounded by C

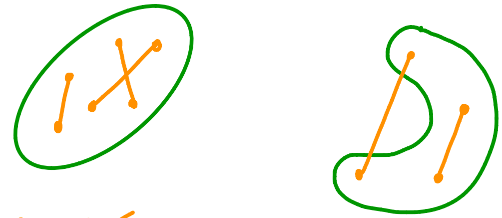
Remarks:

1) Jordan curve theorem is easier to prove if C is regular
but still difficult Topp's book

2) Schoenflies theorem says U_1 "is" a 2-disk
(more precisely U_1 is homeomorphic to D^2
this means $\exists\, f: U_1 \to D^2$ a continuous bijection with continuous inverse)

Definition: a region R in $\mathbb{R}^n$ is convex if for each pair of points $\vec{p}, \vec{q} \in R$ the line segment between them is also in R

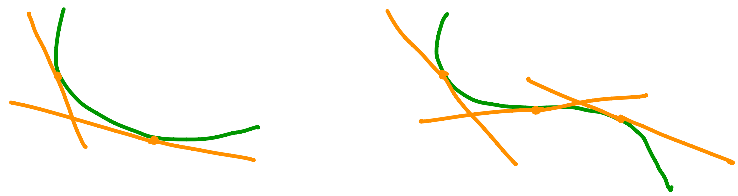
examples:



convex   not convex

we call a regular plane curve C convex if it lies on one side of each of its tangent lines

examples:



convex   not convex

given a regular curve C of length ℓ the total curvature is

$$\tau k(C) = \int_0^\ell \kappa(s)\,ds = \int_0^\ell |\kappa_\sigma(s)|\,ds$$

Theorem 12:

For a simple closed regular curve C in $\mathbb{R}^2$
The following are equivalent:

1) signed curvature of C doesn't change sign — pointwise info
2) $\tau k(C) = 2\pi$ — integral info
3) C is a convex curve — tangent lines
4) the region bounded by C is convex — region bounded by curve

Proof:

1) ⇒ 2): Since κ_σ has constant sign $\kappa(s) = \begin{cases} \kappa_\sigma(s) \\ -\kappa_\sigma(s) \end{cases}$ for all s

thus $\tau k(C) = \int_0^{\ell \leftarrow \text{length of } C} \kappa(s)\,ds = \begin{cases} \int_0^\ell \kappa_\sigma(s)\,ds & \text{if } \kappa_\sigma(s) \geq 0 \\ -\int_0^\ell \kappa_\sigma(s)\,ds & \text{if } \kappa_\sigma(s) \leq 0 \end{cases}$

$= |TK(C)| = 2\pi$ ← corollary 11 ✓

2) $\Rightarrow$ 1):

$$2\pi = \tau k(C) = \int_0^\ell \kappa(s)\,ds = \int_0^\ell |\kappa_\sigma(s)|\,ds$$

$$\overset{\circledast}{\geq} \left|\int_0^\ell \kappa_\sigma(s)\,ds\right| = 2\pi$$

corollary 11

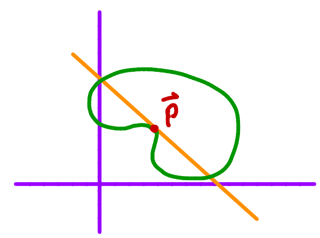
$\circledast$ $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$ with equality $\Leftrightarrow$ f doesn't change sign

$\therefore$ $\kappa_\sigma(s)$ doesn't change sign ✓

1) $\Rightarrow$ 3): by 1) we know $\kappa_\sigma(s)$ has constant sign

to show 3) we assume it is false and derive a contradiction

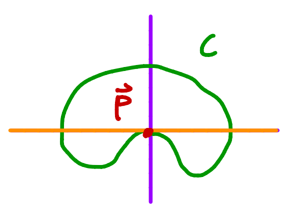
so let $\vec{p}$ be a point on C such that C doesnot lie on one side of $T_{\vec{p}}C$



we can move C by a rigid motion so $\vec{p}$ is (0.0) and $T_{\vec{p}}C$ = x-axis

(recall this follows from an exercise from I.B)

so we have



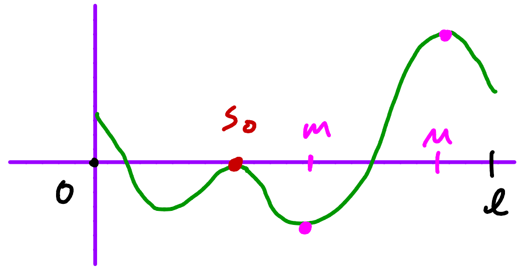
now let $\vec{\beta}:[0,\ell]\to\mathbb{R}^2$ be an arc length parameterization of C

note: there is some $s_0\in[0,\ell]$ such that

$$\vec{\beta}(s_0) = (0.0)$$

$$\vec{\beta}\,'(s_0) = \begin{bmatrix}1\\0\end{bmatrix} \text{ or } \begin{bmatrix}-1\\0\end{bmatrix}$$

let $\pi:\mathbb{R}^2\to\mathbb{R}:(x,y)\mapsto y$ be projection to y-axis so $f = \pi \circ \vec{\beta} : [0,\ell] \to \mathbb{R}$ has graph



$f(s_0) = 0$

chain rule

$$f'(s_0) = \left(D\pi_{\vec{\beta}(0)}\right)\vec{\beta}\,'(s_0)$$

$$= [0, 1]\begin{bmatrix}\pm 1\\ 0\end{bmatrix} = 0$$

recall: since f is continuous it takes on a

a maximum at some point M and

a minimum " " m

and $f'(M) = 0 = f'(m)$

<u>exercise</u>: if m, M are on interior of $[0,\ell]$ this

last statement clear from calculus

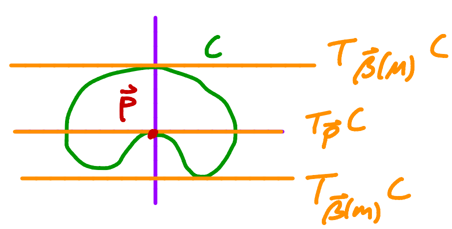
if not, then show this is still true

<u>hint</u>: $\vec{\beta}\,'(0) = \vec{\beta}\,'(\ell)$

also note $f(m) < 0 < f(M)$ or else C would be

above or below the x-axis $= T_{\vec{p}}C$

so $T_{\beta(M)}C \neq T_{\vec{p}}C \neq T_{\beta(m)}C$



<u>note</u>: at least two of

$\vec{\beta}\,'(m), \vec{\beta}\,'(M), \vec{\beta}\,'(s_0)$ must

be the same

(only 2 unit vectors on x-axis)

suppose $\vec{\beta}\,'(m) = \vec{\beta}\,'(M)$ (you can check other cases work the same)

recall there is a function $\theta : [0,\ell] \to \mathbb{R}$

such that $\vec{\beta}\,'(s) = (\cos\theta(s), \sin\theta(s))$

note: a) C a simple curve implies (by Corollary 11) that

$$R(C) = \frac{1}{2\pi}\left(\theta(\ell) - \theta(0)\right)$$
$$= \pm 1$$

by choice of orientation we can assume $+1$

(note when you change or$^{\underline{n}}$ you change signed curvature)

so $\boxed{\theta(\ell) = \theta(0) + 2\pi}$

b) We know $\theta'(s) = \kappa_o(s)$

so θ' doesn't change sign (with or$^{\underline{n}}$ above it will be non-negative)

$$\boxed{\theta'(s) \geq 0}$$

c) since we have $\vec{\beta}'(m) = \vec{\beta}'(M)$ we know

$$\boxed{\theta(m) = \theta(M) + n2\pi}$$

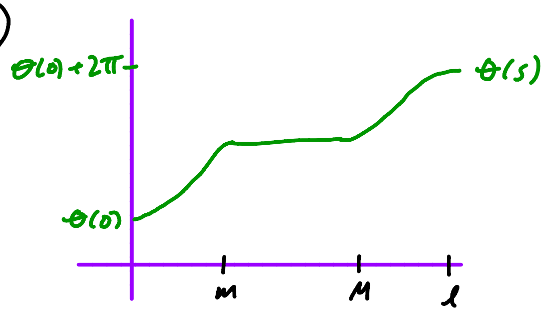
for some integer n

a) and b) $\Rightarrow$ θ is non-decreasing going from $\theta(0)$ to $\theta(0) + 2\pi$

so for all $s \in [0, \ell]$ we have $\theta(s) \in [\theta(0), \theta(0) + 2\pi]$

thus either $\theta(M) = \theta(m)$ I or $\theta(M) = \theta(m) \pm 2\pi$ II

for I



so $\theta(s) = \theta(m)$ for all s between m and M

so $\vec{\beta}'(s)$ is constant for all s between m and M

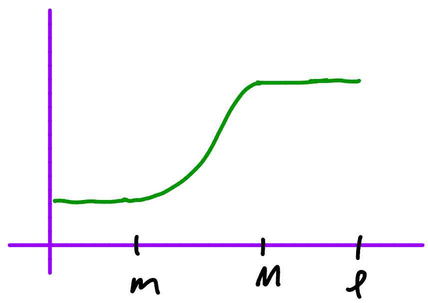
that means all points on C "between" $\beta(m)$ and $\beta(M)$

lie on a line in $\mathbb{R}^2$

and thus $T_{\vec{\beta}(m)} C = T_{\beta(M)} C$

contradicting our observation above!

for II you can make an argument as above that the graph of θ is



so still get $T_{\vec{\beta}(m)} C = T_{\vec{\beta}(M)} C$

a contradiction

thus 3) must be true ✓

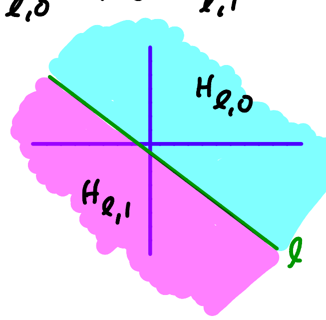
3) $\Rightarrow$ 4):

exercise:

1) If $\{R_\alpha\}_{\alpha \in A}$ is a collection of convex regions indexed by A, then $\bigcap_{\alpha \in A} R_\alpha$ is convex

2) let ℓ be a line in $\mathbb{R}^2$ and $H_{\ell,0}$ and $H_{\ell,1}$ the "half-spaces" ℓ divides $\mathbb{R}^2$ into

Show $H_{\ell,0}$, $H_{\ell,1}$ are convex

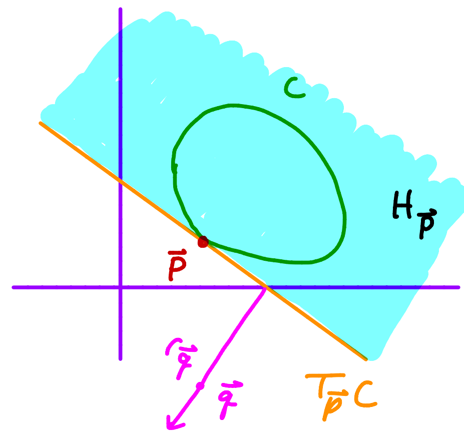


$H_{\ell,0} \cap H_{\ell,1} = \ell$

now let D be the region C bounds in $\mathbb{R}^2$

for each $\vec{p} \in C$ the hypothesis 3) says C is contained in one of the half spaces given by $T_{\vec{p}} C$

denote it $H_{\vec{p}}$



note: $D \subset H_{\vec{p}}$ too

indeed if $\vec{q} \notin H_{\vec{p}}$

then let $\vec{r_q}$ be the ray (half-line) through $\vec{q}$ and perpendicular to $T_{\vec{p}}C$

$\vec{r_q}$ gives a "path to ∞" for $\vec{q}$ in the complement of C so $\vec{q}$ is in the unbounded component of $\mathbb{R}^2 - C$

thus not in D

$\therefore D \subset H_{\vec{q}}$

so we see $\underbrace{D \subset \bigcap_{\vec{p} \in C} H_{\vec{p}}}_{\text{denote this } R}$

note R is convex by exercise 1) and 2)

Claim: $R \subset D$ (note this establishes 4) since $D = R$ is convex)

indeed, if $\vec{q} \notin D$ then let

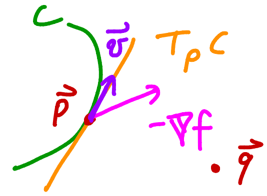
$f(\vec{x}) = \|\vec{q} - \vec{x}\|^2$ = square of distance from $\vec{x}$ to $\vec{q}$

f is a differentiable function and hence continuous

thus there is a $\vec{p} \in C$ s.t.

$f(\vec{p}) \leq f(\vec{r})$ for all $\vec{r} \in C$

note: $\nabla f(\vec{p}) \perp T_{\vec{p}} C$ since if not



$-\nabla f$ is direction of steepest decent

so if $\vec{v}$ tanget to C, at $\vec{p}$ then $-\nabla f \cdot \vec{v} \neq 0$ and f can decreace as you move along C in direction of $\vec{v}$ $\otimes$ $\vec{p}$ being a min

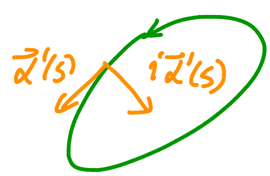
now $\vec{q} \notin H_{\vec{p}}$

thus $\vec{q} \notin R$ ✓

<u>4) $\Rightarrow$ 1):</u> It is clear that for some orientation on C we can arc length parameterize C by

$$\vec{\alpha} : [0, \ell] \to \mathbb{R}^2$$

such that $i\vec{\alpha}'(s)$ points <u>into</u> the region D bounded by C



Claim: $\kappa_\sigma(s) \geq 0$ for all s ($\therefore$ 1) is true)

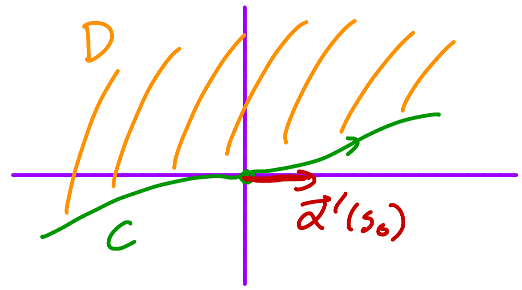
to see this suppose $\kappa_\sigma(s_0) < 0$ for some s_0

we can move C by a rigid motion so that

$$\vec{\alpha}(s_0) = (0,0) \quad \text{and} \quad \vec{\alpha}'(s_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(<u>note</u>: this does not change curvature or convexity of D) prove this!

so we see near (0,0)



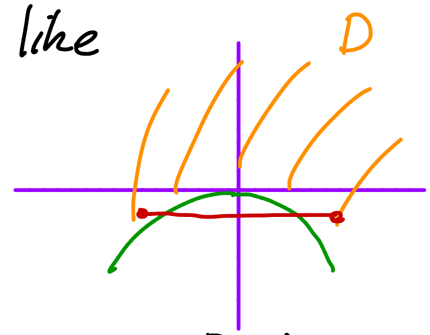
Claim: near (0,0) C is the graph of a function

$$f:(-\varepsilon_1,\varepsilon_2)\longrightarrow\mathbb{R}$$

with $f(0)=0$, $f'(0)=0$ and $\kappa_\sigma(0)=f''(s_0)$

given claim we see f is concave down at s_0 ($f''(0)<0$)

so C looks like



and we see D is not convex! $\emptyset$

$\therefore\ \kappa_\sigma(s)\geq 0$ for all s i.e. 1) true

Proof of Claim:

let $g:[0,\ell]\to\mathbb{R}$ be given by $g(s)=\pi_x(\vec{\alpha}(s))$

where $\pi_x:\mathbb{R}^2\to\mathbb{R}:(x,y)\mapsto x$

then $g'(s_0)=D(\pi_x)_{\vec{\alpha}(s_0)}\cdot\vec{\alpha}'(s_0)$

$$=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=1$$

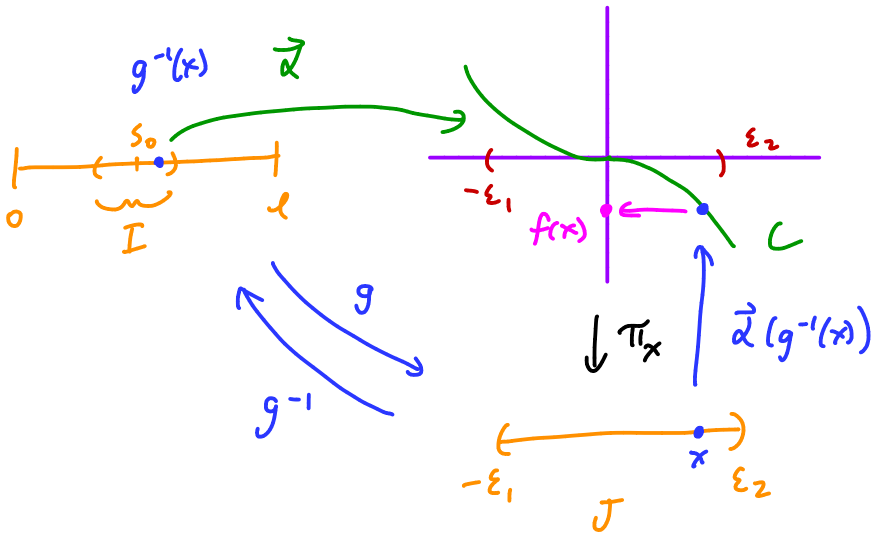
so the inverse function theorem says there are intervals $I=(-\varepsilon_1,\varepsilon_2)\subset\mathbb{R}$ $\varepsilon_2>0$

$J=(s_0-\delta,s_0+\delta)\subset[0,\ell]$ $\delta>0$

such that $\exists$ an inverse $g^{-1}:J\to I$

now set $f: J \to \mathbb{R} : x \mapsto \pi_y(\vec{\alpha}(g^{-1}(x)))$

where $\pi_y : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto y$



so

$\vec{\alpha}(g^{-1}(x)) = (x, f(x))$

and so C near (0,0) is the grapf of a function f

$$f'(x) = D(\pi_y)_{\vec{\alpha}(g^{-1}(x))}(\vec{\alpha}'(g^{-1}(x)))\,(g^{-1})'(x)$$ chain rule

$$= [0\ 1](\vec{\alpha}'(g^{-1}(x)))\ \frac{1}{g'(g^{-1}(x))}$$ inverse function theorem

$$= [0\ 1](\vec{\alpha}'(g^{-1}(x)))\ \frac{1}{D(\pi_x)_{\vec{\alpha}(g^{-1}(x))}\,\vec{\alpha}'(g^{-1}(x))}$$ chain rule

$$= \left[[0\ 1]\ \vec{\alpha}'(g^{-1}(x))\right]\left[\frac{1}{[1\ 0]\ \vec{\alpha}'(g^{-1}(x))}\right]$$

$$f''(x) = \left[[0\ 1]\ \vec{\alpha}''(g^{-1}(x))\,(g^{-1})'(x)\right]\left[\frac{1}{[1\ 0]\,\vec{\alpha}'(g^{-1}(x))}\right]$$ product + chain rule

$$+\left[[0\ 1]\ \vec{\alpha}'(g^{-1}(x))\right]\left[\frac{1}{[1\ 0]\,\vec{\alpha}''(g^{-1}(x))(g^{-1})'(x)}\right]$$

$$f''(0) = \left[[0\ 1]\ \vec{\alpha}''(s_0)\,(g^{-1})'(0)\right]\frac{1}{[1\ 0]\begin{bmatrix}1\\0\end{bmatrix}}$$

$$+\underbrace{\left[[0\ 1]\begin{bmatrix}1\\0\end{bmatrix}\right]}_{0}\left[\sim\sim\right]$$

$$= \left[\underbrace{i\,\vec{\alpha}'(s_0)}_{\begin{bmatrix}0\\1\end{bmatrix}}\right]\cdot\vec{\alpha}''(s_0)\,(g^{-1})'(0) = \kappa_\sigma(s_0)\ \frac{1}{g'(s_0)}$$

$$= \kappa_\sigma(s_0)$$

Th$^{\underline{m}}$ 13 (Fenchel in $\mathbb{R}^2$):

for any regular simple closed curve in $\mathbb{R}^2$

$$\tau K(C) \geq 2\pi$$

with equality iff C is convex

Proof:

$$\tau K(C) = \int_0^\ell K(s)\,ds = \int_0^\ell |K_\sigma(s)|\,ds \geq \left|\int_0^\ell K_\sigma(s)\,ds\right|$$

$$= 2\pi\,|R(C)| = 2\pi$$

and we have equality $\Leftrightarrow$ $K_\sigma(s)$ does not change sign

thus we are done by Th$^{\underline{m}}$ 12 ▦

Given any function $f: [0,\ell] \to \mathbb{R}$ is it the signed curvature function of a closed convex curve?

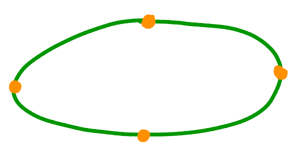
No! need 1) f does not change sign

2) $\int_0^\ell f(s)\,ds = \pm 2\pi$

It turns out there is another restriction too

we call a point $\vec{p}$ on a closed curve C a vertex if $K_\sigma'(\vec{p}) = 0$

example:



4 verticies

Th$^{\underline{m}}$ 14 (Four-Vertex Theorem):

Every simple closed regular convex curve in $\mathbb{R}^2$ has at least 4 verticies

for a proof see any book

4. Length, width, and Area

Thm 15 (Isoparametric Inequality):

> let C be a simple closed regular curve in $\mathbb{R}^2$
> D be the region in $\mathbb{R}^2$ bounded by C
>
> Then
> $$\text{Area}(D) \leq \frac{1}{4\pi}(\text{length}(C))^2$$
> with equality $\Leftrightarrow$ C a circle

Interpretation: If you have a fixed amount of fencing and you want to enclose as much area as possible then build a circular fence!

Proof: lots of different proof of this
we give one based on Fourier Series

"Recall": if $f:[0,\ell]\to\mathbb{R}$ is a smooth function
(with $f^{(n)}(0)=f^{(n)}(\ell)$ $n=0,1,2$)

then set
$$\left.\begin{aligned} a_n &= \frac{2}{\ell}\int_0^\ell f(x)\cos\frac{n2\pi}{\ell}x\,dx \\ b_n &= \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n2\pi}{\ell}x\,dx \end{aligned}\right\}\text{Fourier coefficiants}$$

then
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi n}{\ell}x+b_n\sin\frac{2\pi n}{\ell}x\right)$$

exercise: the coefficients a_n' and b_n' of f' are

$$a_n'=\frac{2\pi n}{\ell}b_n \quad\text{and}\quad b_n'=-\frac{2\pi n}{\ell}a_n$$

if c_n and d_n are the coefficients of g, then

$$\int_0^\ell f(x)g(x)\,dx=\frac{\ell}{2}\left(\frac{a_0c_0}{2}+\sum_{n=1}^{\infty}a_nc_n+b_nd_n\right)$$

now suppose $\vec{\alpha}:[0,\ell]\to\mathbb{R}^2$ is an arc length parameterization of C (oriented counterclockwise)

let $\vec{\alpha}(s)=(f(s),g(s))$

and a_n, b_n be the Fourier coeff of f

and c_n, d_n " " g

by translation we can assume $a_0=0=c_0$

then

$$\underbrace{\text{length}(C)}_{\ell} = \int_0^\ell \|\vec{\alpha}'(s)\|\,ds = \int_0^\ell \sqrt{(f'(s))^2+(g'(s))^2}\,ds$$

$\|\vec{\alpha}'(s)\|=1$

$$= \int_0^\ell (f'(s))^2+(g'(s))^2\,ds$$

$$= \frac{\ell}{2}\left[\left(\frac{2\pi}{\ell}\right)^2\sum_{n=1}^\infty n^2\left\{(a_n^2+b_n^2)+(c_n^2+d_n^2)\right\}\right]$$

$$\text{so}\quad \frac{\ell^2}{2\pi^2} = \sum_{n=1}^\infty n^2(a_n^2+b_n^2+c_n^2+d_n^2)$$

now

Green's Th^{m} calc III

$$\text{Area}(D) = \int_D dA = \int_D dx\,dy = \int_C x\,dy$$

$$= \int_0^\ell f(s)g'(s)\,ds$$

$$= \frac{\ell}{2}\left[\frac{2\pi}{\ell}\sum_{n=1}^\infty n(a_nd_n - b_nc_n)\right]$$

$$= \pi\left|\sum_{n=1}^\infty n(a_nd_n-b_nc_n)\right|$$

since area is positive

$$\overset{(*)}{\le} \pi\sum_{n=1}^\infty n\left(|a_n||d_n|+|b_n||c_n|\right)$$

<u>note</u>: $0\le(a-b)^2 = a^2+b^2-2ab$

so $ab \le \frac{1}{2}(a^2+b^2)$

with equality $\Leftrightarrow$ $a=b$

thus
$$\text{Area } D \underset{(**)}{\leq} \frac{\pi}{2}\sum_{n=1}^{\infty} n(a_n^2+b_n^2+c_n^2+d_n^2)$$
$$\underset{(***)}{\leq} \frac{\pi}{2}\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2+c_n^2+d_n^2)$$

$n \leq n^2$ (with $= \iff n=1$)

$$= \frac{\pi}{2}\frac{\ell^2}{2\pi^2} = \frac{\ell^2}{4\pi} \text{ as claimed!}$$

if we have equality then $(*)$, $(**)$, and $(***)$ must all be equalities

$(***)$ an equality $\iff$ $a_n = b_n = c_n = d_n = 0 \quad \forall n > 1$

$(**)$ an equality $\iff$ $|a_1| = |d_1|$, $|b_1| = |c_1|$

$(*)$ an equality $\iff$ $\text{sgn}(a_1 d_1) = -\text{sgn}(b_1 c_1)$

$\therefore$ either $a_1 = d_1$ and $b_1 = -c_1$

or $a_1 = -d_1$ and $b_1 = c_1$

we consider first case and leave second case as <u>exercise</u>

so $f(s) = a_1 \cos\frac{2\pi}{\ell}s + b_1 \sin\frac{2\pi}{\ell}s$

$g(s) = -b_1 \cos\frac{2\pi}{\ell}s + a_1 \sin\frac{2\pi}{\ell}s$

we want to show this parameterizes a circle

note $\ell = \int_0^\ell \|\vec{\alpha}'(s)\| ds = \sqrt{2}\,\pi\, 2\sqrt{a_1^2+b_1^2}$

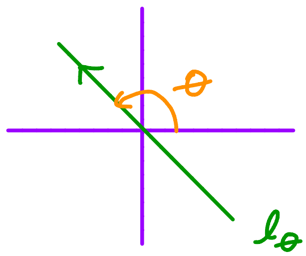
so $a_1^2+b_1^2 = \frac{\ell^2}{8\pi^2}$

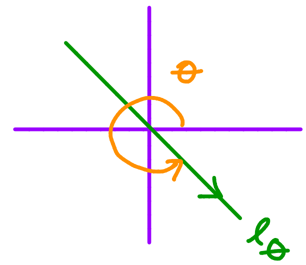
$\therefore \exists\, \theta$ s.t. $a_1 = \underbrace{\frac{\ell^2}{8\pi^2}}_{\text{denote by } K} \sin\theta \qquad b_1 = \frac{\ell^2}{8\pi^2}\cos\theta$

so $f(s) = K\left(\sin\theta\cos\frac{2\pi}{\ell}s + \cos\theta\sin\frac{2\pi}{\ell}s\right) = K\sin\left(\frac{2\pi}{\ell}s+\theta\right)$

and $g(s) = K\left(-\cos\theta\cos\frac{2\pi}{\ell}s + \sin\theta\sin\frac{2\pi}{\ell}s\right) = -K\cos\left(\frac{2\pi s}{\ell}+\theta\right)$

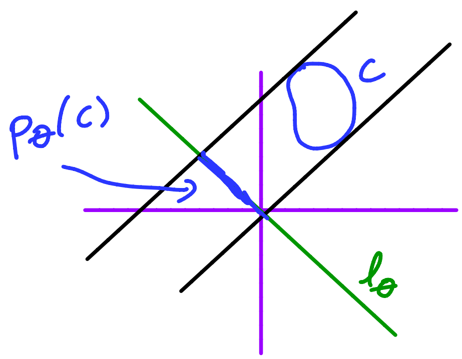
so clearly $\|\vec{\alpha}(s)\| = f^2(s)+g^2(s) = K$ and $\vec{\alpha}$ param. a circle let θ be an angle and l_θ the oriented line through the origin making an angle θ with the positive x-axis





projection to l_θ is the map

$$P_\theta : \mathbb{R}^2 \to \mathbb{R}$$
$$\vec{v} \mapsto \vec{v}_\theta \cdot \vec{v} \quad \text{where } \vec{v}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ spans } l_\theta$$



for a curve C the projection $P_\theta(C)$ to l_θ is the "shadow" of C on l_θ

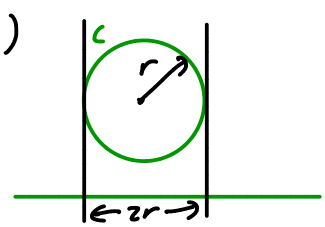
<u>note</u>: $P_\theta(C)$ is some interval on l_θ

Thm 16 (Cauchy-Crofton formula):

> let C be a curve in $\mathbb{R}^2$
> Then
> $$\text{length}(C) = \frac{1}{4}\int_0^{2\pi} \text{length}(P_\theta(C))\, d\theta$$

this says the length of a curve is the average length of its projections onto all lines through origin $\times$ a constant factor

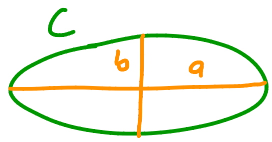
<u>example</u>: 1)



for circle of radius r all projections have length $2r$

but $P_\theta(C)$ has length $4r$ since goes across shadow twice

$$\text{so length } C = \frac{1}{4}\int_0^{2\pi} 4r \, d\theta$$
$$= \frac{1}{4}\, 4r(2\pi - 0) = 2\pi r \checkmark$$

2) 

ellipse

length = ?

but notice all projections have length $\geq 2b$

$$\text{so length}(C) = \frac{1}{4}\int_0^{2\pi} \text{length } P_\theta(C)\, d\theta$$
$$\geq \frac{1}{4}\int_0^{2\pi} 2b \, d\theta$$
$$= b\pi$$

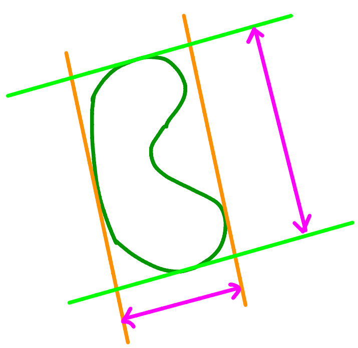
so we can estimate length!

Corollary 17:

$$\boxed{\begin{array}{l}\text{for any closed curve } C \\ \qquad \text{width}(C) \leq \dfrac{\text{length } C}{\pi}\end{array}}$$

the width of C is the minimal distance between parallel lines that contain C

example:



Proof: all projections have length ≥ 2 width ∎

Proof of Thm 16:

let $\vec{\alpha}: [0, \ell] \to \mathbb{R}^2$ be an arc length parameterization of C

recall $p_\theta \circ \vec{\alpha}(s) = \vec{v}_\theta \cdot \vec{\alpha}(s)$ where $\vec{v}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ spans ℓ_θ

so $\text{length}(P_\theta(c)) = \text{length}(\rho_\theta \circ \vec{\alpha})$

$$= \int_0^{\ell} \|(\vec{v}_\theta \cdot \vec{\alpha}(s))'\| \, ds$$

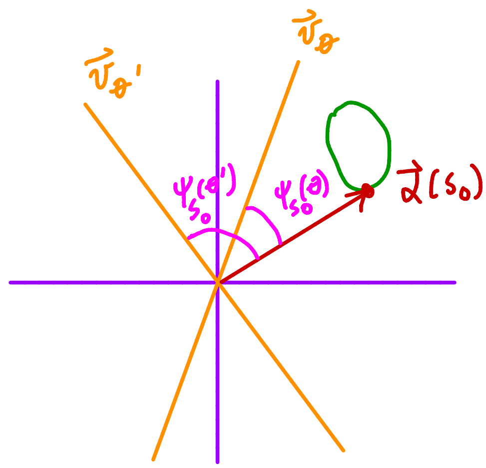
$$= \int_0^{\ell} \|\vec{v}_\theta \cdot \vec{\alpha}'(s)\| \, ds$$

$$= \int_0^{\ell} \underbrace{\|\vec{v}_\theta\|}_{1} \, \underbrace{\|\vec{\alpha}'(s)\|}_{1} \, |\cos \psi_s| \, ds$$

$$= \int_0^{\ell} |\cos \psi_s| \, ds$$

where ψ_s is the angle between $\vec{v}_\theta$ and $\vec{\alpha}'(s)$

<u>note:</u> for a fixed s_0 as θ goes from 0 to 2π, ψ_{s_0} varies between 0 and 2π



and

$$\psi_{s_0}(\theta) = \theta + \theta_{s_0}$$

some fixed θ_{s_0}

so $\displaystyle\int_0^{2\pi} \text{length}(\rho_\theta(c)) \, d\theta = \int_0^{2\pi} \left(\int_0^{\ell} |\cos \psi_s(\theta)| \, ds \right) d\theta$

$$= \int_0^{\ell} \left(\int_0^{2\pi} |\cos \psi_s(\theta)| \, d\theta \right) ds$$

$$= \int_0^{\ell} \left(\int_0^{2\pi} |\cos(\theta_{s_0} + \theta)| \, d\theta \right) ds$$

$$= \int_0^{\ell} \left(\int_{-\frac{\pi}{2} - \theta_{s_0}}^{\frac{\pi}{2} - \theta_{s_0}} \cos(\theta_{s_0} + \theta) \, d\theta - \int_{\frac{\pi}{2} - \theta_{s_0}}^{\frac{3\pi}{2} - \theta_{s_0}} \cos(\theta_{s_0} + \theta) \, d\theta \right) ds$$

$$= \int_0^{\ell} \left(\sin \tfrac{\pi}{2} - \sin -\tfrac{\pi}{2} - \sin \tfrac{3\pi}{2} + \sin \tfrac{\pi}{2} \right) ds$$

$$= \int_0^{\ell} 4 \, ds = 4\ell$$