C. Curves in $\mathbb{R}^3$

let $\vec{\alpha}: [0, \ell] \to \mathbb{R}^3$ be an arc length parameterization of some curve C

Recall: 1) $\|\vec{\alpha}'(s)\| = 1$

2) $\vec{\alpha}''(s)$ is perpendicular to $\vec{\alpha}'(s)$ (lemma 2)

3) we set
$$\vec{N}(s) = \frac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|} \qquad (\text{if } \|\vec{\alpha}''(s)\| \neq 0)$$
and called it the normal vector

4) $\kappa(s) = \|\vec{\alpha}''(s)\|$ is the curvature of C and measures how far C is from being a line segment

5) $\vec{T}(s) = \vec{\alpha}'(s)$ and $\vec{N}(s)$ span a plane
we call this plane the osculating plane
this plane is the plane C comes closest to lying in

6) given two vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

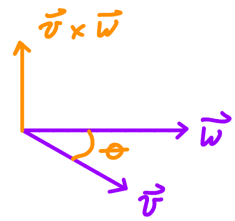
their cross product is

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\vec{i} - (v_1 w_3 - w_1 v_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$$

where $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

so
$$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

and $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$ and $\vec{w}$



"right hand rule"

and $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$

now define $\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$

this is the binormal vector to C

note: $\|\vec{B}(s)\| = \|\vec{T}(s)\| \|\vec{N}(s)\| \sin\frac{\pi}{2} = 1$

so $\vec{B}(s)$ is a unit vector perpendicular to $\vec{T}(s)$ and $\vec{N}(s)$

We define the torsion of C at $\vec{\alpha}(s_0)$ to be

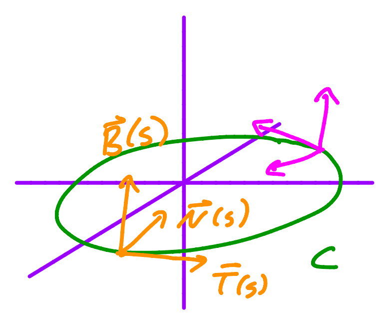
$$\tau(s) = -\vec{B}'(s) \cdot \vec{N}(s)$$

examples:

1) $\vec{\alpha}(s) = (\cos s, \sin s, 0)$

$\vec{T}(s) = \vec{\alpha}'(s) = (-\sin s, \cos s, 0)$

$\vec{\alpha}''(s) = (-\cos s, -\sin s, 0)$



$\vec{N}(s) = \dfrac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|} = (-\cos s, -\sin s, 0)$

so $\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin s & \cos s & 0 \\ -\cos s & -\sin s & 0 \end{bmatrix}$

$$= 0\,\vec{i} - 0\,\vec{j} + (\sin^2 s + \cos^2 s)\,\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\vec{B}'(s) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\tau(s) = 0$

2) Earlier we computed the arc length parameterization of a helix as

$$\vec{\alpha}(s) = \left(r\cos\frac{1}{\sqrt{r^2+b^2}}s,\ r\sin\frac{1}{\sqrt{r^2+b^2}}s,\ \frac{bs}{\sqrt{r^2+b^2}}\right)$$

$$\vec{T}(s) = \left(\frac{-r}{\sqrt{r^2+b^2}}\sin\frac{s}{\sqrt{r^2+b^2}},\ \frac{r}{\sqrt{r^2+b^2}}\cos\frac{s}{\sqrt{r^2+b^2}},\ \frac{b}{\sqrt{r^2+b^2}}\right)$$

$$\vec{T}'(s) = \frac{r}{r^2+b^2}\left(-\cos\frac{s}{\sqrt{r^2+b^2}},\ -\sin\frac{s}{\sqrt{r^2+b^2}},\ 0\right)$$

so $\kappa(s) = \dfrac{|r|}{r^2+b^2}$ constant

and $\vec{N}(s) = \left(-\cos\dfrac{s}{\sqrt{r^2+b^2}},\ -\sin\dfrac{s}{\sqrt{r^2+b^2}},\ 0\right)$

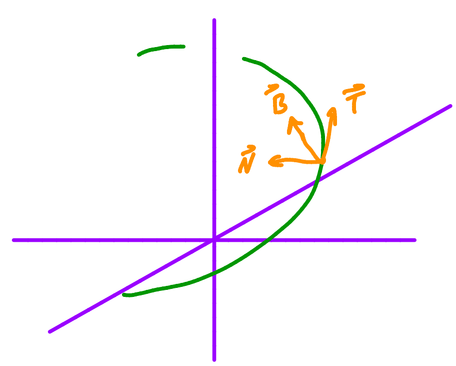
so

$$\vec{B}(s) = \vec{T}(s)\times\vec{N}(s) = \left(\frac{b}{\sqrt{r^2+b^2}}\sin\frac{s}{\sqrt{r^2+b^2}},\ \frac{-b}{\sqrt{r^2+b^2}}\cos\frac{s}{\sqrt{r^2+b^2}},\ \frac{r}{\sqrt{r^2+b^2}}\right)$$

and

$$\vec{B}'(s) = \frac{b}{r^2+b^2}\left(\cos\frac{s}{\sqrt{r^2+b^2}},\ \sin\frac{s}{\sqrt{r^2+b^2}},\ 0\right)$$

so $\tau(s) = -\vec{B}'(s)\cdot\vec{N}(s) = \dfrac{-b}{r^2+b^2}$ constant



exercise: given any two constants $k>0$, τ there is some choice of r and b s.t.

$$k = \frac{|r|}{r^2+b^2}$$

and

$$\tau = \frac{-b}{r^2+b^2}$$

Th$^{\underline{m}}$ 18 (Frenet Formula):

let $\vec{\alpha}:[0,\ell] \to \mathbb{R}^3$ be an arc length parameterization of C

assume $\kappa(s) > 0$. Then

$$\vec{T}'(s) = \kappa(s)\,\vec{N}(s)$$

$$\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s)$$

$$\vec{B}'(s) = -\tau(s)\,\vec{N}(s)$$

Proof: the first equation is the definition of $\vec{N}(s)$ and $\kappa(s)$

note: for each s, $\vec{T}(s), \vec{N}(s), \vec{B}(s)$ is an orthonormal basis for $\mathbb{R}^3$ so any vector is a linear comination

e.g. $\vec{B}'(s) = a\vec{T}(s) + b\vec{N}(s) + c\vec{B}(s)$

by the definition of torsion

$$\begin{aligned} -\tau(s) &= \vec{B}'(s)\cdot\vec{N}(s) \\ &= (a\vec{T}(s) + b\vec{N}(s) + c\vec{B}(s))\cdot\vec{N}(s) \\ &= b \end{aligned}$$

recall $\vec{B}\cdot\vec{T} = 0$ so

$$\vec{B}'\cdot\vec{T} + \vec{B}\cdot\vec{T}' = 0$$

and

$$a = \vec{B}'\cdot\vec{T} = -\vec{B}\cdot\vec{T}' = -\vec{B}\cdot\kappa(s)\,\vec{N}(s) = 0$$

also $\vec{B}\cdot\vec{B} = 1$ so

$$\vec{B}'\cdot\vec{B} + \vec{B}\cdot\vec{B}' = 0$$

and

$$c = \vec{B}\cdot\vec{B}' = 0$$

thus we see $\vec{B}'(s) = -\tau(s)\,\vec{N}(s)$

similarly $\vec{N}'(s) = a\vec{T}(s) + b\vec{N}(s) + c\vec{B}(s)$

$\vec{N}\cdot\vec{N} = 1 \Rightarrow \vec{N}\cdot\vec{N}' = 0$ so as above $b = 0$

$$\vec{N}\cdot\vec{T}=0 \Rightarrow \vec{N}'\cdot\vec{T}+\vec{N}\cdot\vec{T}'=0$$

$$\text{so } a=\vec{N}'\cdot\vec{T}=-\vec{N}\cdot\vec{T}'=-\vec{N}\cdot\chi(s)\vec{N}=-\chi(s)$$

$$\vec{B}\cdot\vec{N}=0 \Rightarrow c=\vec{B}\cdot\vec{N}'=-\vec{B}'\cdot\vec{N}=\tau(s)$$

$$\text{so } \vec{N}'(s)=-\chi(s)\vec{T}(s)+\tau(s)\vec{N}(s)$$

Thm 19:

> If C is a biregular curve (i.e. $\chi(s)\neq 0$) then C lies in a plane $\Leftrightarrow \tau(s)=0$

Proof:

let P_s = plane spanned by $\vec{N}(s)$ and $\vec{T}(s)$ (osculating plane)

since $\vec{B}(s)=\vec{T}(s)\times\vec{N}(s)$, $\vec{B}(s)$ is perpendicular to P_s

so if $\vec{B}(s)$ is consthant then the osculating plane is constant

$\vec{B}'(s)=-\tau(s)\vec{N}(s)$ so osculating plane is constant $\Leftrightarrow \tau(s)=0$

now ($\Rightarrow$) if C lies in a plane then $\vec{T}$ and $\vec{N}$ are always span this fixed plane so $\vec{B}(s)$ is constant $\therefore \vec{B}'(s)=0$ and so $\tau(s)=0$

($\Leftarrow$) if $\tau(s)=0$ then $\vec{B}'(s)=0$ so the plane spanned by $\vec{N}(s)$ and $\vec{T}(s)$ is constant

exercise: if $\vec{T}(s)$ is always in a fixed plane then so is C.

Hint: use isometry to make plane xy-plane then consider the z-coordinate of a parameterization ▦

Thm 20 (Fundamental Theorem of space curves):

given:
1) $I = [0, \ell] \subset \mathbb{R}$
2) $c : I \to \mathbb{R}$ $c(s) > 0$ and differentiable
3) $t : I \to \mathbb{R}$ " "
4) $\vec{p}, \vec{v} \in \mathbb{R}^3$ with $\|\vec{v}\| = 1$

then there exists a unique curve C in $\mathbb{R}^3$ with an arc length parameterization

$$\vec{\alpha} : I \to \mathbb{R}^3$$

such that
1) $\vec{\alpha}(0) = \vec{p}$
2) $\vec{\alpha}'(0) = \vec{v}$
3) $\kappa(s) = c(s)$
4) $\tau(s) = t(s)$

Proof: Uniqueness is an exercise
the existence uses differential equations so we skip it for now (see Do Carmo's Book) ▦

Cor 21:

a biregular curve in $\mathbb{R}^3$ is a helix

$\Leftrightarrow$

τ and κ are constant

Proof: exercise ▦

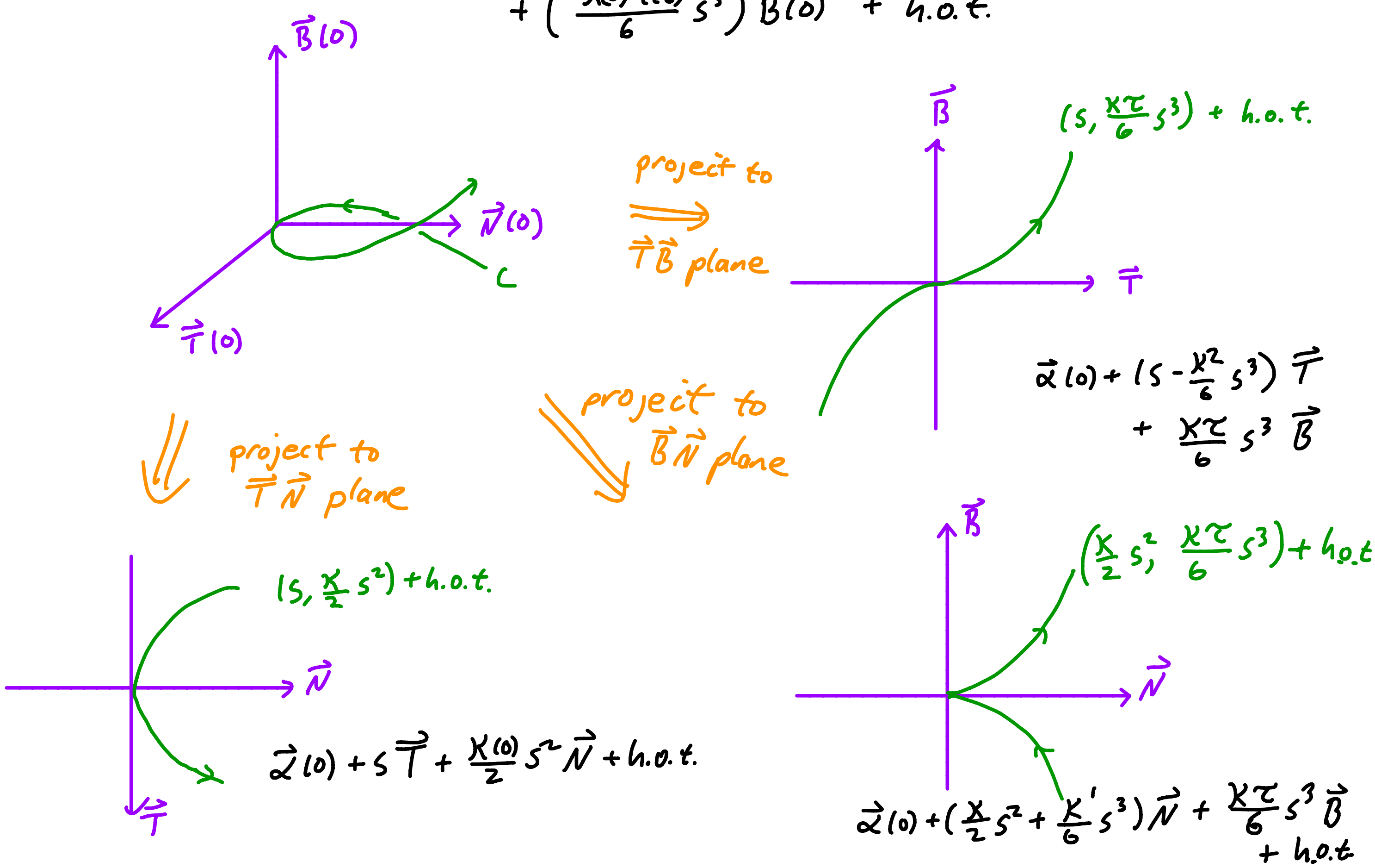
We know for a biregular curve κ and τ essentially determine the curve, but let's see what they really say about the curve

let $\vec{\alpha} : [0, \ell] \to \mathbb{R}^3$ be an arc length parameterization

recall:
$$\vec{\alpha}'(s) = \vec{T}(s)$$
$$\vec{\alpha}''(s) = \kappa(s)\,\vec{N}(s)$$
$$\vec{\alpha}'''(s) = (\kappa(s)\,\vec{N}(s))' = \kappa'(s)\,\vec{N}(s) + \kappa(s)\overbrace{(-\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s))}^{\vec{N}' \text{ by Thm 18}}$$

so the Taylor expansion of $\vec{\alpha}(s)$ about $s=0$ is

$$\begin{aligned}\vec{\alpha}(s) &= \vec{\alpha}(0) + \vec{T}(0)\,s + \frac{\kappa(0)}{2}\vec{N}\,s^2 \\ &\quad + \frac{1}{6}\left[\kappa'(0)\,\vec{N}(0) - (\kappa(0))^2\,\vec{T}(0) + \kappa(0)\tau(0)\,\vec{B}(0)\right]s^3 + \text{higher order terms} \\ &= \vec{\alpha}(0) + \left(s - \frac{(\kappa(0))^2}{6}s^3\right)\vec{T}(0) \\ &\quad + \left(\frac{\kappa(0)}{2}s^2 + \frac{\kappa'(0)}{6}s^3\right)\vec{N}(0) \\ &\quad + \left(\frac{\kappa(0)\tau(0)}{6}s^3\right)\vec{B}(0) + \text{h.o.t.}\end{aligned}$$



exercise:

1) You can think of these projections as shadows take twigs from a tree or bent up paperclips and see you can (almost) always find these three projections/shadows
2) think about what this means about curvature and torsion!

Now for a few global theorems

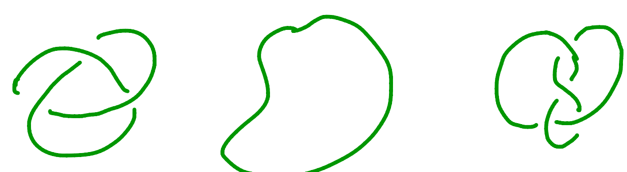
Fenchel's Theorem:

$\boxed{\text{The total curvature of a regular closed curve in } \mathbb{R}^3 \text{ is } \geq 2\pi \quad \text{it is } 2\pi \Leftrightarrow \text{it lies in a plane and is simple and convex}}$

see Tapp's book for a proof

What about other closed curves

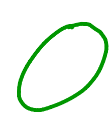
a closed curve C is called a knot if it has an injective parameterization



knots

not a knot

two knots are isotopic (or equivalent) if you can get from one to the other through a continuous family of knots

K is called the unknot if it is isotopic to 

Fary-Milnor Thm:

let K be a knot in $\mathbb{R}^3$
If the total curvature is $< 4\pi$
then K is the unknot