III Manifolds

A. Recollection From Calculus:

let $\vec{f}: \mathbb{R}^m \to \mathbb{R}^n$ be a function

if $\vec{v} \in \mathbb{R}^m$, then the directional derivative of $\vec{f}$ in the direction $\vec{v}$ at $\vec{p} \in \mathbb{R}^m$ is

$$\vec{f}_{\vec{v}}(\vec{p}) = \lim_{h \to 0} \frac{\vec{f}(\vec{p} + h\vec{v}) - \vec{f}(\vec{p})}{h}$$ (if the limit exists)

notation: if $\vec{v} = \frac{\partial}{\partial x_i}$ (a coordinate vector)

then we write $\frac{\partial \vec{f}}{\partial x_i}$ for $\vec{f}_{\vec{v}}$

$\vec{f}$ is differentiable at $\vec{p}$ if there is a linear map

$$B: \mathbb{R}^m \to \mathbb{R}^n$$

such that

$$\frac{\vec{f}(\vec{p} + \vec{v}) - \vec{f}(\vec{p}) - B\vec{v}}{\|\vec{v}\|} \to \vec{0}$$

as $\vec{v} \to \vec{0}$

if B exist we say the (total) derivative of $\vec{f}$ at $\vec{p}$ is B and denote it $(D\vec{f})_{\vec{p}} = B$

Remark: $D\vec{f}_{\vec{p}}$ is the "best linear approximation" to $\vec{f}$ at $\vec{p}$

Calculus Th$^{\underline{m}}$:

1) if $D\vec{f}_{\vec{p}}$ exist then for all $\vec{v} \in \mathbb{R}^m$

$$\underbrace{\vec{f}_{\vec{v}}(\vec{p})}_{\text{directional derivative}} = \underbrace{(D\vec{f})_{\vec{p}}}_{\text{total derivative}}(v)$$

2) if $\vec{f}(\vec{p}) = (f_1(\vec{p}), \ldots, f_n(\vec{p}))$ where $f_i: \mathbb{R}^m \to \mathbb{R}$, then

a) if $\vec{f}$ is differentiable at $\vec{p}$, then the linear map $Df_{\vec{p}}$ expressed using the standard basis in $\mathbb{R}^m$ and $\mathbb{R}^n$ is given by the $n \times m$ matrix

$$(D\vec{f})_p = \left(\frac{\partial f_i}{\partial x_j}(\vec{p})\right)$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{p}) & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{p}) & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

b) if $\frac{\partial f_i}{\partial x_j}(\vec{p})$ exist and are continuous at $\vec{p}$ for all i, j, then $\vec{f}$ is differentiable at $\vec{p}$

examples:

1) $g(x,y) = x^2 + y^2$

so $g : \mathbb{R}^2 \to \mathbb{R}$

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= 2x \\ \frac{\partial g}{\partial y} &= 2y \end{aligned}\right\}$$ exist and are continuous so g is differentiable

$$Dg_{(x,y)} = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

a 1×2 matrix

2) $\vec{f}(x,y) = (x, y, x^2 + y^2)$

so $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$

$$Df_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2y \end{pmatrix}$$

Chain Rule

If $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g}: \mathbb{R}^m \to \mathbb{R}^k$ are differentiable functions, then

$$D(\vec{g} \circ \vec{f})_{\vec{p}} = \underbrace{\left[(D\vec{g})_{\vec{f}(\vec{p})}\right]}_{\text{total derivative of } \vec{g} \text{ at } \vec{f}(\vec{p})} \underbrace{(D\vec{f})_{\vec{p}}}_{\text{total derivative of } \vec{f} \text{ at } \vec{p}}$$

Thm 1:

given $\vec{f}: \mathbb{R}^m \to \mathbb{R}^n$

$\vec{v} \in \mathbb{R}^m$ a vector

$\vec{p} \in \mathbb{R}^m$ a point

let $\vec{\alpha}: [a,b] \to \mathbb{R}^n$ be a path such that

$\vec{\alpha}(c) = \vec{p}$ for some $\vec{p} \in (a,b)$

$\vec{\alpha}'(c) = \vec{v}$

then $(D\vec{f})_{\vec{p}}(\vec{v}) = \frac{d}{dt}(\vec{f} \circ \vec{\alpha})\Big|_{t=c}$

Proof:

Chain rule gives

$$\begin{aligned}
\frac{d}{dt}(\vec{f} \circ \vec{\alpha})(c) &= (\vec{f} \circ \vec{\alpha})_{\frac{d}{dt}}(c) \\
&= D(\vec{f} \circ \vec{\alpha})_c\left(\frac{d}{dt}\right) \\
&= (D\vec{f}_{\vec{\alpha}(c)})(D\vec{\alpha}_c)\frac{d}{dt} \\
&= (D\vec{f}_{\vec{p}})(\vec{\alpha}_{\frac{d}{dt}}(c)) \\
&= (D\vec{f}_{\vec{p}})(\vec{\alpha}'(c)) \\
&= (D\vec{f}_{\vec{p}})\vec{v}
\end{aligned}$$

B. Manifolds in $\mathbb{R}^n$

a (regular) k-manifold, or manifold of dimension k, M is a subset of $\mathbb{R}^n$ (for some n) such that for each point $\vec{p} \in M$ there is

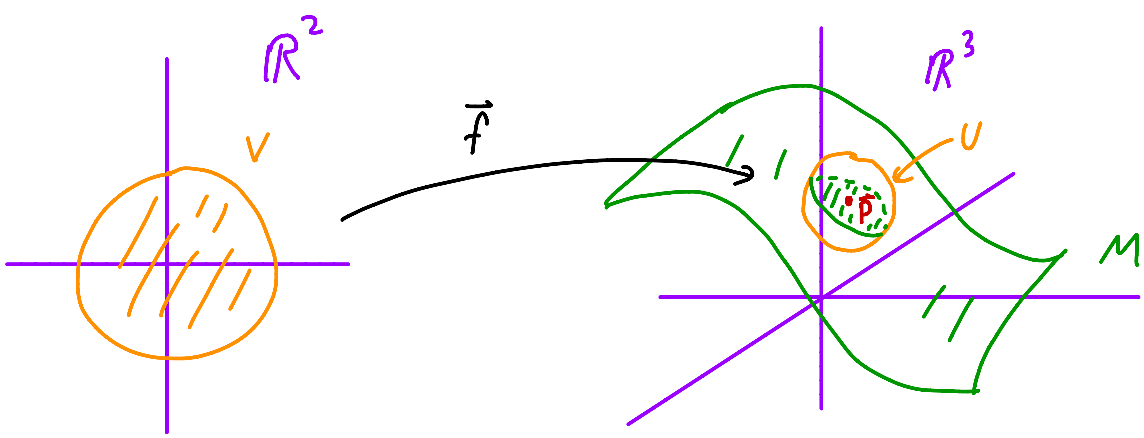
1) an open set U in $\mathbb{R}^n$ about $\vec{p}$,
2) an open set V in $\mathbb{R}^k$, and
3) a differentiable map

$$\vec{f} : V \to U$$

such that

a) $\vec{f}$ is injective
b) $\text{image}(\vec{f}) = M \cap U$
c) $\text{Rank } D\vec{f}_{\vec{q}} = k$ for all $\vec{q} \in V$
d) $\vec{f}^{-1} : (U \cap M) \to V$ is continuous

we will usually not worry about this too much
frequently automatic



$\vec{f}$ is called a local parameterization or a coordinate chart around $\vec{p}$

Idea: if you "live" in M then it seems like you are in $\mathbb{R}^k$ (if you can't see too far!)

exercise: a simple regular closed curve C in $\mathbb{R}^n$ is a 1-manifold

a 2-manifold is also called a surface

a manifold $M \subset \mathbb{R}^n$ is called closed if the set M is closed (i.e. for every sequence $\{x_n\}$ in M that converges to some point $x_n \to x$ in $\mathbb{R}^n$ we have $x \in M$)

we say M is compact if it is closed and bounded in $\mathbb{R}^n$

(this is equivalent to saying any sequence in M has a subsequence that converges to a point in M)

examples:

1) let P be a plane in $\mathbb{R}^n$

so there are vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ and a point $\vec{p} \in \mathbb{R}^n$ such that

$$P = \{ \vec{p} + a\vec{v}_1 + b\vec{v}_2 \mid a, b \in \mathbb{R} \}$$

note:

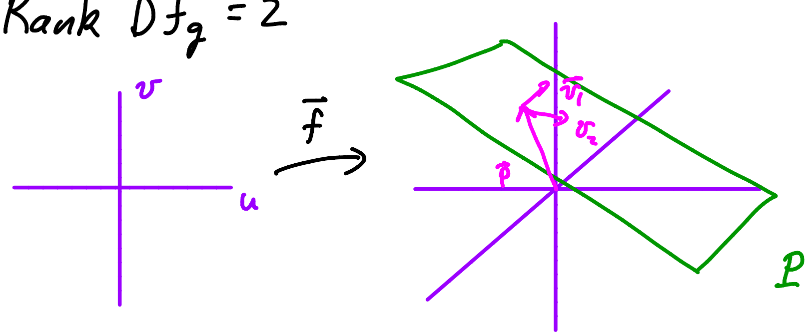
$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^n : (u,v) \longmapsto \vec{p} + u\vec{v}_1 + v\vec{v}_2$

is a local parameterization for any point in P

since: $\vec{f}$ clearly injective and onto P, and

$$D\vec{f}_{\vec{q}} = [\vec{v}_1 \; \vec{v}_2] \quad \text{for all } \vec{q}$$

so $\operatorname{Rank} D\vec{f}_q = 2$



note: 1) ℓ is a closed surface

2) any k-space in $\mathbb{R}^n$ is a k-manifold

2) let $g: \mathbb{R}^2 \to \mathbb{R}^1$ be smooth function

$$\Gamma_g = \{ (\vec{p}, g(\vec{p})) \mid \vec{p} \in \mathbb{R}^2 \}$$ graph of g

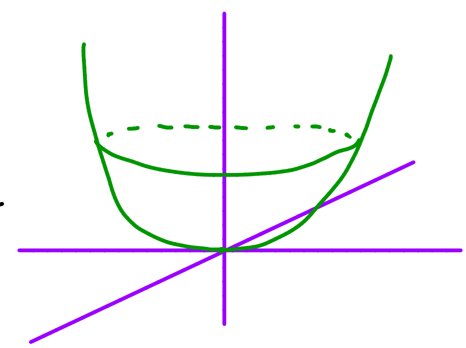
is a surface in $\mathbb{R}^3$ because

$$\vec{f}(u,v) = (u, v, g(u,v))$$

gives a local param. since it is clearly injective and onto, and

$$Df = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ g_u & g_v \end{bmatrix}$$ has rank 2

e.g. $g(x,y) = x^2 + y^2$



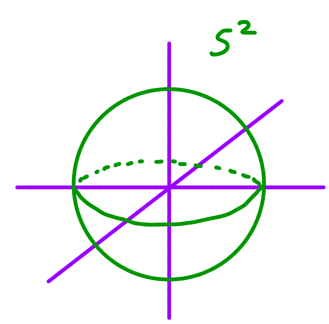
note: $\vec{g}: \mathbb{R}^k \to \mathbb{R}^{n-k}$ a smooth function

gives a k-manifold in $\mathbb{R}^n$

$$\Gamma_{\vec{g}} = \{ (\vec{p}, \vec{g}(\vec{p})) \mid \vec{p} \in \mathbb{R}^k \}$$

3) let $S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$

2-sphere



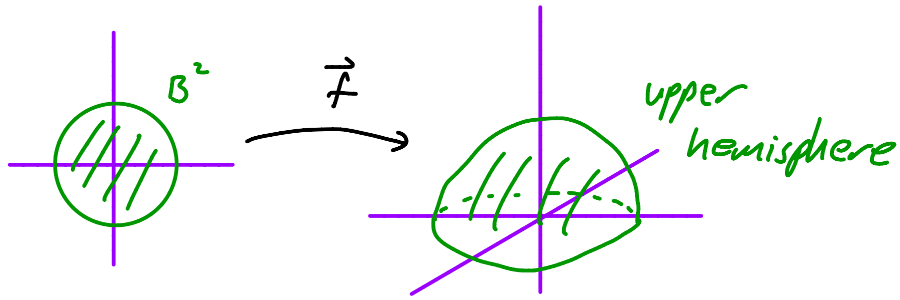
Claim: S^2 is a compact surface

to see this suppose $\vec{p} \in S^2$

and z-coord of $\vec{p} > 0$

then let $B^2 = \{ (u,v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1 \}$

$$\vec{f}: B^2 \to \mathbb{R}^3 : (u,v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$$



$$D\vec{f}_{(u,v)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{u}{\sqrt{}} & \frac{v}{\sqrt{}} \end{bmatrix} \qquad \text{rank } 2$$

if x-coord $\vec{p} > 0$ then use

$$\vec{f}(u,v) = \left(\sqrt{1-u^2-v^2},\, u,\, v\right)$$

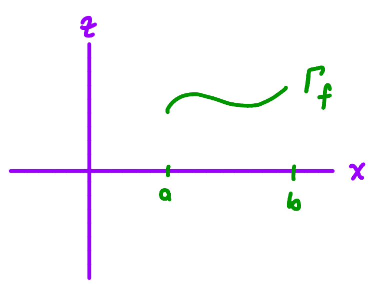
you can find 6 such local param. that cover every point in S^2

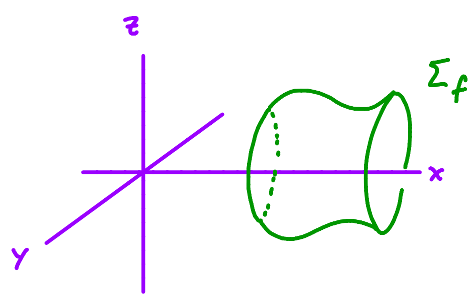
exercise: 1) S^2 is compact

2) $S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1 \}$ is a compact n-manifold

4) Surfaces of revolution

given a graph $f : (a,b) \to \mathbb{R}$ in the xz-plane rotate it about the x-axis





$$\Gamma_f = \{ (x, f(x)) \mid x \in (a,b) \} \subset \mathbb{R}^2$$

or

$$= \{ (x, 0, f(x)) \mid x \in (a,b) \} \subset \mathbb{R}^3$$

surface of revolution

$$\Sigma_f = \{ (x, f(x)\cos\theta, f(x)\sin\theta) \mid x \in (a,b),\ \theta \in [0, 2\pi] \}$$

define $g : (a,b) \times \mathbb{R} \to \mathbb{R}^3$

$$(x, \theta) \longmapsto (x, f(x)\cos\theta, f(x)\sin\theta)$$

exercise: Show g restricted to subsets of $(a,b)\times\mathbb{R}$ give local parameterizations of Σ_f

let $M \subset \mathbb{R}^n$ be a k-manifold and

$\vec{g} : M \to \mathbb{R}^m$ be a function

we say $\vec{g}$ is differentiable (or smooth) at $\vec{p} \in M$

if there is a coordinate chart

$$\vec{f} : V \to U, \quad V \subset \mathbb{R}^k, \quad U \subset \mathbb{R}^n$$

such that $\vec{p} \in U \cap M$ (so $\exists\, \vec{q} \in V$ s.t. $\vec{f}(\vec{q}) = \vec{p}$)

and

$$\vec{g} \circ \vec{f} : V \to \mathbb{R}^m$$

is differentiable at $\vec{q}$

we say $\vec{g}$ is differentiable if it is differentiable at all points of M

if $M' \subset \mathbb{R}^{n'}$ is a k'-manifold and

$g : M \to M'$ a function

we say g is differentiable if $g : M \to \mathbb{R}^{n'}$ is

If $M \subset \mathbb{R}^n$ is a k-manifold, then we say a vector $\vec{v} \in \mathbb{R}^n$

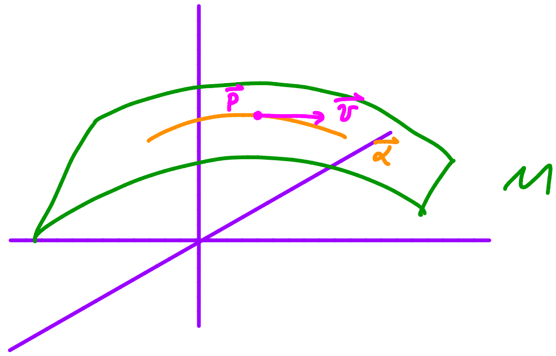
is tangent to M at $\vec{p} \in M$ if there is a regular curve

$$\vec{\alpha} : [a,b] \to \mathbb{R}^n$$

such that 1) $\operatorname{Im}\vec{\alpha} \subset M$

2) $\vec{\alpha}(c) = \vec{p}$ some $c \in (a,b)$

3) $\vec{\alpha}'(c) = \vec{v}$

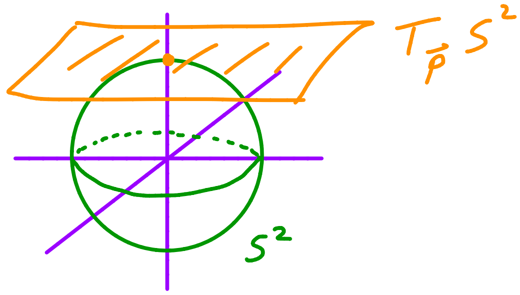


The <u>tangent space to M at $\vec{p}$</u> is

$$T_{\vec{p}} M = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \text{ is tangent to } M \text{ at } \vec{p}\}$$

<u>example:</u> $S^2 \subset \mathbb{R}^3$

$\vec{p} = (0,0,1)$



$\vec{f}(u,v) = (u, v, \sqrt{1-u^2-v^2})$ gives coordinate chart near $\vec{p} = \vec{f}(0,0)$

$$D\vec{f}_{\vec{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ vector in $\mathbb{R}^2$ then

$\vec{\beta}(t) = t\vec{v}$ a path in $\mathbb{R}^2$

$\vec{\alpha} = \vec{f} \circ \vec{\beta}$ a path in $\mathbb{R}^3$ such that

image$(\vec{\alpha}) \subset S^2$

$$\vec{\alpha}'(0) = (D\vec{f}_{\vec{\beta}(0)})\, \vec{\beta}'(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$$

so we see that any vector in the xy-plane is "tangent to S^2 at $(0,0,1)$"

that is $\quad$ xy-plane $\subset T_{(0,0,1)} S^2$

to see the other inclusion let $\vec{v} \in T_{(0,0,1)} S^2$

by definition there is some path

$$\vec{\gamma} : (-\varepsilon, \varepsilon) \to S^2$$

such that $\vec{\gamma}(0) = (0,0,1)$ and $\vec{\gamma}'(0) = \vec{v}$

if ε is small enough, then $\operatorname{im} \vec{\gamma} \subset \operatorname{im} \vec{f}$

if $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x,y)$ is projection

then $\vec{\alpha} = \pi \circ \vec{\gamma}$ is a curve in $\mathbb{R}^2$ such that

$\vec{\gamma} = \vec{f} \circ \vec{\alpha}$

thus if $\vec{v}\,' = \vec{\alpha}\,'(0)$ we see

$$D\vec{f}_{(0,0)}(\vec{v}\,') = \vec{v}$$

so $\vec{v} \in$ image $D\vec{f}_{(0,0)} =$ xy-plane

and $T_{(0,0,1)} S^2 =$ "xy-plane"

<u>exercise:</u> 1) if $\vec{f} : V \to U$ is a coordinate chart ($V \subset \mathbb{R}^k$) for $M \subset \mathbb{R}^n$ about $\vec{p} \in M$

then
$$T_{\vec{p}} M = \operatorname{Image}(D\vec{f}_{\vec{q}})$$
$$= \operatorname{span}\{\vec{f}_{\vec{e}_1}(\vec{q}), \ldots, \vec{f}_{\vec{e}_k}(\vec{q})\}$$

where $\vec{e}_1, \ldots, \vec{e}_k$ is a basis for $\mathbb{R}^k$ and $\vec{q} \in V$ such that $\vec{f}(\vec{q}) = \vec{p}$

2) M a k-manifold, then $T_{\vec{p}} M$ a k-dim'l vector space