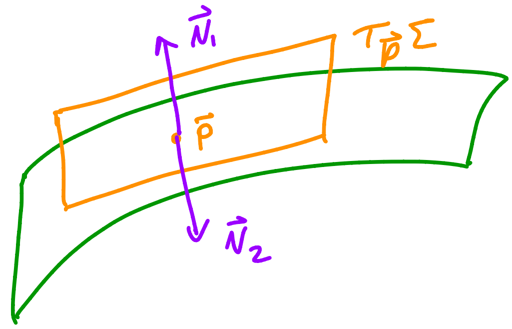
# IV Surfaces

## A. Geometry of Surfaces in $\mathbb{R}^3$

let $\Sigma^2 \subset \mathbb{R}^3$ be a regular surface

$\vec{p} \in \Sigma$

note: $T_{\vec{p}}\Sigma$ is a plane in $\mathbb{R}^3$ so there are two unit normal vectors in $\mathbb{R}^3$ perpendicular to $T_{\vec{p}}\Sigma$




e.g. if $\vec{v}_1, \vec{v}_2$ are two linearly indep vectors in $T_{\vec{p}}\Sigma$ then

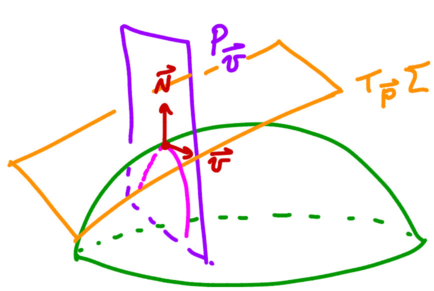
$$\pm\vec{N} = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|}$$

are normal vectors

Choose one normal vector $\vec{N}$

for any unit vector $\vec{v} \in T_{\vec{p}}\Sigma$ we define the curvature of $\Sigma$ at $\vec{p}$ in the direction $\vec{v}$ as follows:

let $P_{\vec{v}}$ = plane at $\vec{p}$ spanned by $\vec{v}$ and $\vec{N}$




near $\vec{p}$, $P_{\vec{v}} \cap \Sigma$ is a curve $C$

let $\vec{\beta}: [-\varepsilon, \varepsilon] \to \Sigma$ parameterize $C$ such that

$$\|\vec{\beta}\,'(s)\| = 1$$
$$\vec{\beta}(0) = \vec{p}$$
$$\vec{\beta}\,'(0) = \vec{v}$$

note: $C \subset P_{\vec{v}}$ so $\vec{\beta}\,'(0) \in P_{\vec{v}}$ and $\vec{\beta}\,''(0) \in P_{\vec{v}}$

also $\vec{\beta}\,''(0)$ perpendicular to $\vec{\beta}\,'(0)$ (lemma II, 2)

so $\vec{\beta}\,''(0)$ is parallel to $\vec{N}$

$\therefore \exists$ a number $\kappa_{\vec{p}}(\vec{v}) \in \mathbb{R}$ such that

$$\vec{\beta}\,''(0) = \kappa_{\vec{p}}(\vec{v})\,\vec{N}$$

(i.e. $\pm\kappa_{\vec{p}}(\vec{v})$ is the signed curvature of $C$ in the plane $P_{\vec{v}}$)

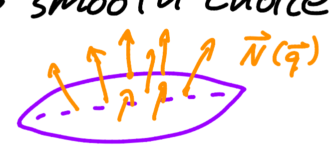
we call $\kappa_{\vec{p}}(\vec{v})$ the <u>curvature of $\Sigma$ at $\vec{p}$ in the direction $\vec{v}$</u>

lemma 1:

> let $\vec{\gamma}: [a,b] \to \Sigma$ be any curve in $\Sigma$ with
> $$\|\vec{\gamma}\,'(s)\| = 1,$$
> $$\vec{\gamma}(c) = \vec{p} \quad (\text{some } c \in (a,b)), \text{ and}$$
> $$\vec{\gamma}\,'(c) = \vec{v}$$
> then $\kappa_{\vec{p}}(\vec{v}) = \vec{\gamma}\,''(c) \cdot \vec{N}$

(so you can use <u>any</u> arc length $\vec{\gamma}$ to define curvature)

Proof: near $\vec{p}$ let $\vec{N}(\vec{q})$ be a smooth choice of unit normal vector for $\vec{q}$ near $\vec{p}$



now let $\vec{\beta}$ be the curve above and $\vec{\gamma}$ as in lemma

so $\vec{\beta}(0) = \vec{\gamma}(c)$ and $\vec{\beta}'(0) = \vec{\gamma}'(0)$

note

$\vec{N}(\vec{\gamma}(s)) \cdot \vec{\gamma}'(s) = 0$ since $\gamma'(s)$ is tangent to $\Sigma$)

so

$$\left[\vec{N}(\vec{\gamma}(s))\right]' \cdot \vec{\gamma}'(s) + \vec{N}(\vec{\gamma}(s)) \cdot \vec{\gamma}''(s) = 0$$

$$\left[(D\vec{N}_{\vec{\gamma}(s)})(\vec{\gamma}'(s))\right] \cdot \vec{\gamma}'(s) + \vec{N}(\vec{\gamma}(s)) \cdot \vec{\gamma}''(s) = 0$$

at $s = c$ we have

$$-\left[(D\vec{N}_{\vec{p}})(\vec{v})\right] \cdot \vec{v} = \vec{N}(\vec{p}) \cdot \vec{\gamma}''(c)$$

similarly

$$-\left[(D\vec{N}_{p})(\vec{v})\right] \cdot \vec{v} = \vec{N}(\vec{p}) \cdot \vec{\beta}''(0) = \kappa_{\vec{p}}(\vec{v})$$

so $\kappa_{\vec{p}}(\vec{v}) = \vec{N}(\vec{p}) \cdot \vec{\gamma}''(c)$

**Remark:** recall for any $\vec{\gamma}(s)$

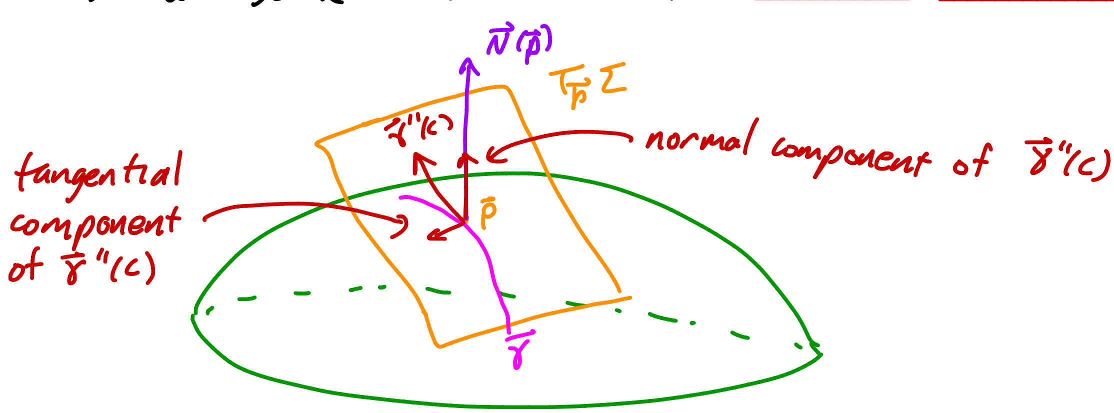
$$\vec{\gamma}''(s) = \underbrace{\kappa(s)}_{\text{curvature of } \vec{\gamma} \text{ at } s} \widetilde{N}(s)$$

$\dfrac{\vec{\gamma}''(s)}{\|\vec{\gamma}''(s)\|}$ normal to $\vec{\gamma}$ at $s$

(previously denoted $\vec{N}(s)$ but that now denotes normal to $\Sigma$ so rename $\widetilde{N}(s)$)

so $\kappa_{\vec{p}}(\vec{v})$ is the component of $\vec{\gamma}''(c)$ in the direction of the normal direction to the surface

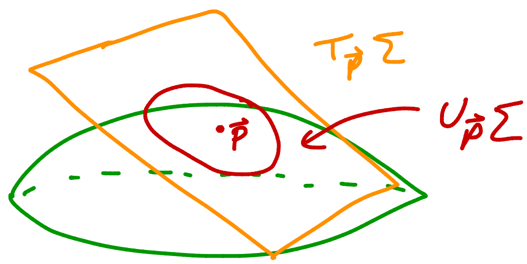
so we sometimes call it the normal curvature



now for $\vec{p} \in \Sigma$ let

$$U_p\Sigma = \{\vec{v} \in T_{\vec{p}}\Sigma \mid \|\vec{v}\| = 1\}$$

circle in $T_{\vec{p}}\Sigma$




we can think of $\mathcal{K}_p$ as a function

$$\begin{tikzcd}
\mathcal{K}_p : U_{\vec{p}}\Sigma \arrow[r] & \mathbb{R} : \vec{v} \longmapsto \mathcal{K}_{\vec{p}}(\vec{v}) \\
S^1 \arrow[u, equal] & \\
{[0,2\pi]} \arrow[u] \arrow[uur, bend right, "\mathcal{K}_{\vec{p}}"'] &
\end{tikzcd}$$

$$(\cos\theta, \sin\theta) \longmapsfrom \theta$$

$\mathcal{K}_p$ is continuous

from calculus we know such a function always has a global maximum and global minimum

let $\mathcal{K}_1 = \max \mathcal{K}_{\vec{p}}$
$\mathcal{K}_2 = \min \mathcal{K}_{\vec{p}}$ } called the principal curvatures at $\vec{p}$

examples:

1) let $g(x,y) = x^2+y^2$

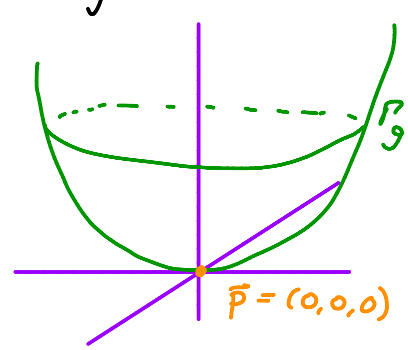
and $\Gamma_g$ = graph of $g$ $= \{(x,y,g(x,y))\}$

let's compute the curvature at $(0,0,0) \in \Gamma_g$

need to choose a normal $\vec{N} = \frac{\partial}{\partial z}$

note $T_{\vec{p}}\Gamma_g$ = xy-plane

notation by $\frac{\partial}{\partial z}$ we mean the vector $\begin{bmatrix}0\\0\\1\end{bmatrix}$



so for any $(v_1, v_2, 0) \in T_{\vec{p}}\Gamma_g$

with $v_1^2 + v_2^2 = 1$

we have $\vec{\beta}(t) = (tv_1, tv_2, (v_1^2+v_2^2)t^2)$

$= (v_1 t, v_2 t, t^2)$

is a curve in $\Gamma_g$

note $\vec{\beta}$ is not an arc length parameterization

think about why this is true

but normal to $\vec{\beta}$ at $(0.0.0)$ is $\frac{\partial}{\partial z}$ so the normal curvature is just the curvature of $C = \mathrm{im}\,\vec{\beta}$ at $C$

from homework we know

$$\kappa(0) = \left\| \left(\frac{\vec{\beta}'(t)}{\|\vec{\beta}'(t)\|}\right)' \frac{1}{\|\vec{\beta}'(t)\|} \right\| \Bigg|_{t=0}$$

$$= \left\| \left(\frac{(v_1, v_2, 2t)}{\sqrt{1+4t^2}}\right)' \frac{1}{\sqrt{1+4t^2}} \right\| \Bigg|_{t=0}$$

$$= \left\| \left[(0,0,2)\frac{1}{\sqrt{1+4t^2}} + (v_1, v_2, 2t)\left(-\tfrac{1}{2}\right)\frac{8t}{(1+4t^2)^{3/2}}\right] \frac{1}{\sqrt{1+4t^2}} \right\| \Bigg|_{t=0}$$

$$= \|(0,0,2) - (0,0,0)\| = 2$$

so $\kappa_{\vec{p}}(\vec{v}) = 2 \quad \forall \vec{v} \in U_{\vec{p}}\Gamma_g$

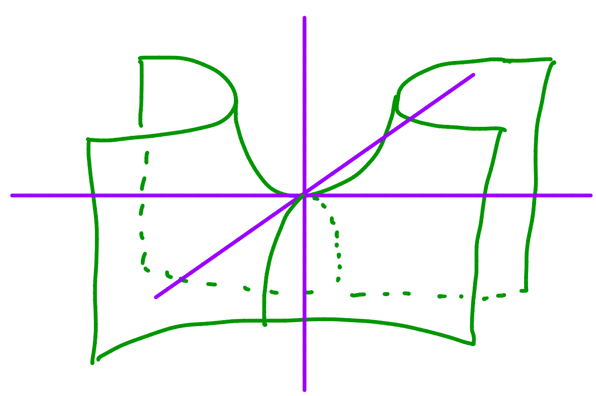
and $\kappa_1 = 2 = \kappa_2$

2) let $f(x,y) = -x^2 + y^2$

$\Gamma_f = \{(x,y,f(x,y))\}$
graph of $f$

$T_{(0,0,0)}\Gamma_f = xy\text{-plane}$

take $\vec{N}_{(0,0,0)} = \frac{\partial}{\partial z}$



you can compute

$$\kappa_{\vec p}\left(\frac{\partial}{\partial x}\right) = -2$$

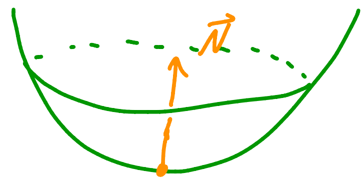
$$\kappa_{\vec p}\left(\frac{\partial}{\partial y}\right) = 2$$

So what does $\kappa_p$ tell us in general?

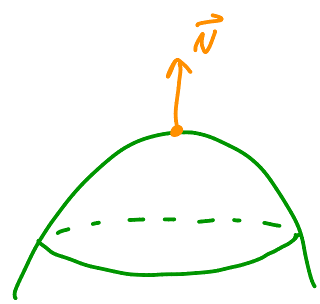
note: 1) if you switch $\vec N$ to $-\vec N$ then $\kappa_{\vec p}$ changes sign

2) $\kappa_{\vec p}(\vec v)$ tells you whether or not $\Sigma$ in the direction of $\vec v$ is bending toward $\vec N$ or away from $\vec N$

3) so if $\kappa_1$ and $\kappa_2$ have the same sign then $\kappa_p$ has a constant sign and $\Sigma$ is always bending towards $\vec N$ or away from $\vec N$

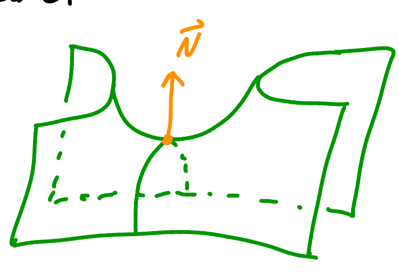



$\kappa_1, \kappa_2 > 0$




$\kappa_1, \kappa_2 < 0$

if $\kappa_1$ and $\kappa_2$ have opposite signs then $\Sigma$ is sometimes bending toward $\vec N$ and sometimes away




$\kappa_1 > 0 > \kappa_2$

so $\kappa_1$ and $\kappa_2$ tell us a lot about how $\Sigma$ locally looks!

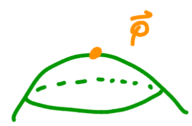
we define the Gauss Curvature of $\Sigma$ at $\vec{p}$ to be

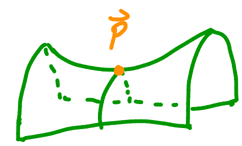
$$K(\vec{p}) = \kappa_1 \kappa_2$$

and the mean curvature to be

$$H(\vec{p}) = \frac{1}{2}(\kappa_1 + \kappa_2)$$

note: $K(\vec{p})$ is independent of which unit normal we take!

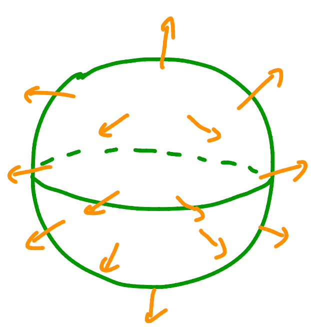
and $K(\vec{p}) > 0 \Rightarrow$ 

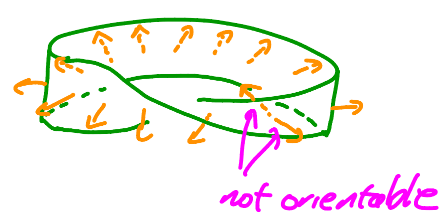
$K(\vec{p}) < 0 \Rightarrow$ something like 

let's reinterperate these curvatures

at each point of a regular surface $\Sigma \subset \mathbb{R}^3$ we can pick a normal vector $\vec{N}$, if we can do this continuously for all $\vec{p} \in \Sigma$ then we say $\Sigma$ is orientable (and a choice of $\vec{N}(\vec{p})$ is called an orientation)

example:






$\boxed{\text{we will always assume } \Sigma \text{ is orientable}}$

example: $\Gamma_g = \{(x,y,g(x,y))\} \subset \mathbb{R}^3$ for some $g: \mathbb{R}^2 \to \mathbb{R}$

recall $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3 : (u,v) \mapsto (u,v,g(u,v))$
is a parameterization of $\Gamma_g$
and thus

$$T_{(u,v,g(u,v))}\Gamma_g = \operatorname{Im} D\vec{f}_{(u,v)}$$
$$= \operatorname{span}\left\{\begin{bmatrix}1\\0\\g_u\end{bmatrix}, \begin{bmatrix}0\\1\\g_v\end{bmatrix}\right\}$$

so a normal vector at $(u,v,g(u,v))$ is

$$\begin{bmatrix}1\\0\\g_u\end{bmatrix} \times \begin{bmatrix}0\\1\\g_v\end{bmatrix} \underset{\text{exercise}}{=} \begin{bmatrix}-g_u\\-g_v\\1\end{bmatrix}$$

so we take an orientation on $\Gamma_g$ to be

$$\vec{N}(u,v,g(u,v)) = \frac{-1}{\sqrt{1+g_u^2+g_v^2}}\begin{bmatrix}g_u\\g_v\\-1\end{bmatrix}$$

Remark: $\vec{N}(\vec{p})$ is a unit vector for all $\vec{p} \in \Sigma$ so we can think of $\vec{N}$ as a function

$$\boxed{\vec{N}: \Sigma \to S^2}$$

this is called the Gauss map

the Shape Operator (or Weingarten Map) is no defined to be

$$S_{\vec{p}}(\vec{v}) = -\underbrace{\vec{N}_{\vec{v}}}_{\substack{\text{directional derivative}\\ \text{of } \vec{N} \text{ in direction of } \vec{v}}}(\vec{p}) = -D\vec{N}_{\vec{p}}(\vec{v})$$

lemma 2:

1) $S_{\vec{p}} : T_{\vec{p}}\Sigma \to T_{\vec{p}}\Sigma$ is a linear map

2) $S_{\vec{p}}(\vec{v}) \cdot \vec{v} = K_{\vec{p}}(\vec{v})$ for $\vec{v} \in U_{\vec{p}}\Sigma$

3) $\langle S_{\vec{p}}(\vec{v}), \vec{w} \rangle = \langle \vec{v}, S_{\vec{p}}(\vec{w}) \rangle$

i.e. $S_{\vec{p}}$ is self-adjoint

Proof: For 1

note $\vec{N} \cdot \vec{N} = 1$

so the product rule gives

$$(\vec{N}_{\vec{v}}) \cdot \vec{N} + \vec{N} \cdot (\vec{N}_{\vec{v}}) = (D\,1)\,\vec{v} = 0$$

and we have

$$(\vec{N}_{\vec{v}}) \cdot \vec{N} = 0$$

so $\vec{N}_{\vec{v}}$ is perpendicular to $\vec{N}$

i.e. $-S_{\vec{p}}(\vec{v}) = \vec{N}_{\vec{v}}(\vec{p})$ is in $T_{\vec{p}}\Sigma$

thus $S_{\vec{p}} : T_{\vec{p}}\Sigma \to T_{\vec{p}}\Sigma$

also
$$\begin{aligned} S_{\vec{p}}(a\vec{v} + b\vec{w}) &= -(\vec{N}_{a\vec{v} + b\vec{w}})(\vec{p}) \\ &= -\underbrace{(D\vec{N}_{\vec{p}})}_{\text{Total derivative}}(a\vec{v} + b\vec{w}) \\ &= -a\, D\vec{N}_{\vec{p}}(\vec{v}) - b\, D\vec{N}_{\vec{p}}(\vec{w}) \\ &= -a\, \vec{N}_{\vec{v}}(\vec{p}) - b\, \vec{N}_{\vec{w}}(\vec{p}) \\ &= a\, S_{\vec{p}}(\vec{v}) + b\, S_{\vec{p}}(\vec{w}) \end{aligned}$$

For 2 let $\vec{v} \in U_{\vec{p}}\Sigma$

and $\vec{\beta}$ any arc length param. of a curve such that

$\mathrm{im}(\vec{\beta}) \subset \Sigma$, $\vec{\beta}(0) = \vec{p}$ and $\vec{\beta}'(0) = \vec{v}$

now we know $\vec{\beta}'(0)\cdot\vec{N}(\vec{\beta}(0))=0$ since $\vec{\beta}'(0)\in T_{\vec{\beta}(0)}\Sigma$

thus the product rule gives

$$\vec{\beta}''(s)\cdot\vec{N}(\vec{\beta}(s))+\vec{\beta}'(s)\cdot\frac{d}{ds}\vec{N}(\vec{\beta}(s))=0$$

and the chain rule gives

$$\vec{\beta}''(s)\cdot\vec{N}(\vec{\beta}(s))+\vec{\beta}'(s)\cdot\left[D\vec{N}_{\vec{\beta}(s)}(\vec{\beta}'(s))\right]=0$$

at 0 we get

$$\underbrace{\vec{\beta}''(0)\cdot\vec{N}(\vec{p})}_{K_{\vec{p}}(\vec{v}) \text{ by lemma 1}}-\vec{v}\cdot S_{\vec{p}}(\vec{v})=0$$

For ③ let $\vec{f}:V\to\mathbb{R}^3$ be a coordinate chart for s.t $\vec{f}(0,0)=\vec{p}$

recall
$$T_{\vec{f}(u,v)}\Sigma=\operatorname{span}\{\underbrace{D\vec{f}_{(u,v)}(\tfrac{\partial}{\partial u})}_{\vec{u}(u,v)},\ \underbrace{D\vec{f}_{(u,v)}(\tfrac{\partial}{\partial v})}_{\vec{v}(u,v)}\}$$

<u>note</u>: $\vec{N}(\vec{f}(u,v))\cdot\underbrace{D\vec{f}_{(u,v)}(\tfrac{\partial}{\partial u})}_{\in T_{\vec{f}(u,v)}\Sigma}=0$

recall we denote $\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$ by $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$

so the product rule gives

$$\frac{\partial}{\partial v}\left(\vec{N}(\vec{f}(u,v))\cdot\vec{u}(u,v)\right)=0$$

$$\shortparallel$$

$$D\vec{N}_{\vec{f}(u,v)}(\vec{v}(u,v))\cdot\vec{u}(u,v)+\vec{N}(\vec{f}(u,v))\cdot D\vec{u}_{(u,v)}\tfrac{\partial}{\partial v}$$

at $(u,v)=(0,0)$

$$0=-S_{\vec{p}}(\vec{v})\cdot\vec{u}+\vec{N}(\vec{p})\cdot D\left[D\vec{f}_{(u,v)}(\tfrac{\partial}{\partial u})\right]_{(0,0)}(\tfrac{\partial}{\partial v})$$

similarly

$$S_{\vec{p}}(\vec{u})\cdot\vec{v}=\vec{N}(\vec{p})\cdot D\left[D\vec{f}_{(u,v)}(\tfrac{\partial}{\partial v})\right]_{(0,0)}(\tfrac{\partial}{\partial u})$$

now to see $S_{\vec{p}}(\vec{v})\cdot\vec{u}=S_{\vec{p}}(\vec{u})\cdot\vec{v}$ consider

$\vec{f}(u,v) = (x(u,v), y(u,v), z(u,v))$

and so

$$D\vec{f} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \quad \text{and} \quad D\vec{f}\left(\tfrac{\partial}{\partial u}\right) = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$$

thus

$$D\left(D\vec{f}\left(\tfrac{\partial}{\partial u}\right)\right) = \begin{bmatrix} x_{uu} & x_{uv} \\ y_{uu} & y_{uv} \\ z_{uu} & z_{uv} \end{bmatrix}$$

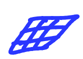
and

$$D\left(D\vec{f}\left(\tfrac{\partial}{\partial u}\right)\right)\left(\tfrac{\partial}{\partial v}\right) = \begin{bmatrix} x_{uv} \\ y_{uv} \\ z_{uv} \end{bmatrix}$$

you can easily check $D\left(D\vec{f}\left(\tfrac{\partial}{\partial v}\right)\right)\left(\tfrac{\partial}{\partial u}\right) =$

and so $S_{\vec{p}}(\vec{u}) \cdot \vec{v} = S_{\vec{p}}(\vec{v}) \cdot \vec{u}$ for the vectors $\vec{u}, \vec{v}$ coming from the coordinate chart

<u>exercise</u>: Use linearity to show 3 true for all vectors in $T_{\vec{p}}\Sigma$



<u>example</u>: $\Gamma_g = \{(x,y,g(x,y))\} \subset \mathbb{R}^3$, $\quad g: \mathbb{R}^2 \to \mathbb{R}$

earlier we saw $\vec{N}(x,y,g(x,y)) = \dfrac{-1}{\sqrt{1+g_x^2+g_y^2}} \begin{bmatrix} g_x \\ g_y \\ -1 \end{bmatrix}$

let's take $g(x,y) = y^2 - x^2$

so $\vec{N}(x,y,g(x,y)) = \dfrac{-1}{\sqrt{1+4x^2+4y^2}} \begin{bmatrix} -2x \\ 2y \\ -1 \end{bmatrix}$

$$S_{(x,y,g(x,y))}(\vec{v}) = -\vec{N}_{\vec{v}}(x,y,g(x,y))$$

let's compute at $(0,0,0)$

recall $T_{(0,0,0)}\Gamma_g = xy\text{-plane}$

given a tangent vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ let

$$\vec{\alpha}(t) = (t v_1, t v_2, (v_2^2 - v_1^2)t^2)$$

be a curve in $\Gamma_g$ s.t. $\vec{\alpha}(0) = (0,0,0)$

$\vec{\alpha}'(0) = (v_1, v_2, 0)$

Thm III.1

$$\text{so } \vec{N}_v(0,0,0) = \frac{\partial}{\partial t}(\vec{N} \circ \vec{\alpha})\Big|_{t=0}$$

$$= \frac{d}{dt}\left( \frac{-1}{\sqrt{1 + 4v_1^2 + 4v_2^2 t^2}} \begin{bmatrix} -2v_1 t \\ 2v_2 t \\ -1 \end{bmatrix} \right)\Bigg|_{t=0}$$

$$= \Bigg( \frac{1}{2} \frac{1}{(\quad)^{3/2}} (4v_1^2 + 4v_2^2) 2t \begin{bmatrix} \quad \end{bmatrix}$$

$$+ \frac{1}{(\quad)^{1/2}} \begin{bmatrix} -2v_1 \\ 2v_2 \\ 0 \end{bmatrix} \Bigg)\Bigg|_{t=0}$$

$$= 0 + \begin{bmatrix} -2v_1 \\ 2v_2 \\ 0 \end{bmatrix}$$

$$\text{so } S_{(0,0,0)}\left(\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}\right) = -\vec{N}_{\vec{v}}(0,0,0) = \begin{bmatrix} 2v_1 \\ -2v_2 \\ 0 \end{bmatrix}$$

let's study the shape operator more

$$S_{\vec{p}} : T_{\vec{p}}\Sigma \to T_{\vec{p}}\Sigma$$

let $\vec{v}, \vec{w}$ be a basis for $T_{\vec{p}}\Sigma$

$$\text{so } S_{\vec{p}}(\vec{v}) = a_{11}\vec{v} + a_{21}\vec{w}$$

$$S_{\vec{p}}(\vec{w}) = a_{12}\vec{v} + a_{22}\vec{w}$$

given <u>any</u> vector $\vec{u} \in T_{\vec{p}}\Sigma$ there are numbers $a, b$ s.t.

$$\vec{u} = a\vec{v} + b\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\text{so } S_{\vec{p}}(\vec{u}) = S_{\vec{p}}(a\vec{v} + b\vec{w}) = a S_{\vec{p}}(\vec{v}) + b S_{\vec{p}}(\vec{w})$$

$$= (a a_{11} + b a_{21})\vec{v} + (a a_{12} + b a_{22})\vec{w}$$

note: $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a a_{11} + b a_{21} \\ a a_{12} + b a_{22} \end{pmatrix}$

so $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$ represents $S_{\vec{p}}$ in the basis $\vec{v}, \vec{w}$

lemma 2, part 3 says $S_{\vec{p}}(\vec{v}) \cdot \vec{w} = S_{\vec{p}}(\vec{w}) \cdot \vec{v}$

so if $\vec{v}, \vec{w}$ are orthonormal, then $a_{21} = a_{12}$

i.e. $A$ is a symmetric matrix

**lemma 3:**

a $2 \times 2$ symmetric matrix has real eigenvalues

**Proof:** find the eigenvalues:

$$\det\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = (a-\lambda)(c-\lambda) - b^2 = 0$$

so $(ac - b^2) - (a+c)\lambda + \lambda^2 = 0$

and
$$\lambda = \frac{a+c \pm \sqrt{(c+a)^2 - 4ac + 4b^2}}{2}$$

$$= \frac{a+c \pm \sqrt{c^2 + a^2 + 2ac - 4ac + 4b^2}}{2}$$

$$= \frac{a+c \pm \sqrt{(c-a)^2 + 4b^2}}{2} \quad \leftarrow \geq 0$$

so there are 2 real eigenvalues

**lemma 4:**

1) eigenvalues of $S_{\vec{p}}$ are the principal curvatures $\kappa_1, \kappa_2$ at $\vec{p}$

2) if $\kappa_1 \neq \kappa_2$, then the eigenvectors $\vec{e}_1, \vec{e}_2$ are perpendicular (we may also take them to be unit length)

3) if $\vec{v}$ is a unit vector, then there is some $\theta$ s.t.

$$\vec{v} = \cos\theta\, \vec{e}_1 + \sin\theta\, e_1$$

and

$$\kappa_{\vec{p}}(\vec{v}) = (\cos^2\theta)\kappa_1 + (\sin^2\theta)\kappa_2$$

so $S_{\vec{p}}$ determines $\kappa_{\vec{p}}$

**Proof:** let $\lambda_1, \lambda_2$ be the eigenvalues of $S_{\vec{p}}$

If $\lambda_1 = \lambda_2$, then $S_{\vec{p}}(\vec{v}) = \lambda_1 \vec{v} \quad \forall \vec{v}$

and by lemma 2 part 2

$$\kappa_{\vec{p}}(\vec{v}) = \lambda_1 \qquad \forall \vec{v} \in U_{\vec{p}}\Sigma$$

and $\kappa_1 = \kappa_2 = \lambda_1 = \lambda_2$ so 1) and 3) true (and 2)!)

If $\lambda_1 \neq \lambda_2$, then let $\vec{e}_1$ be a unit eigenvector for $\lambda_1$

assume $\lambda_1 > \lambda_2$ $\qquad \vec{e}_2$ " $\qquad$ " for $\lambda_2$

note

$$\underset{\lambda_1 \vec{e}_1 \cdot \vec{e}_2}{\underset{\parallel}{(S_{\vec{p}}\vec{e}_1)\cdot\vec{e}_2}} \overset{\text{lemma 2.3}}{=} \underset{\lambda_2 \vec{e}_1 \cdot \vec{e}_2}{\underset{\parallel}{(S_{\vec{p}}\vec{e}_2)\cdot\vec{e}_1}}$$

so $(\lambda_1 - \lambda_2)\,\vec{e}_1 \cdot \vec{e}_2 = 0$

since $\lambda_1 \neq \lambda_2$ we see $\vec{e}_1 \cdot \vec{e}_2 = 0$

now given $\vec{v} \in U_{\vec{p}}\Sigma$ there is a $\theta$ such that

$\vec{v} = (\cos\theta)\vec{e}_1 + (\sin\theta)\vec{e}_2$

now

$$K_{\vec{p}}(\vec{v}) \overset{\text{lemma 2.2}}{=} S_{\vec{p}}(\vec{v}) \cdot \vec{v}$$

$$= S_{\vec{p}}\left((\cos\theta)\vec{e}_1 + (\sin\theta)\vec{e}_2\right) \cdot \left((\cos\theta)\vec{e}_1 + (\sin\theta)\vec{e}_2\right)$$

$$= \left[(\cos\theta)\lambda_1\vec{e}_1 + (\sin\theta)\lambda_2\vec{e}_2\right] \cdot \left[(\cos\theta)\vec{e}_1 + (\sin\theta)\vec{e}_2\right]$$

$$= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$$

but $K_{\vec{p}}(\vec{v}) = \lambda_2 + (\lambda_1 - \lambda_2)\cos^2\theta$

so $K_1 = \max S_{\vec{p}} = \lambda_1$

$K_2 \underset{\text{by def}^{n}}{=} \min S_p = \lambda_2$

thus $K_1, K_2$ are the eigenvalues so 1) is true and 2), 3) follow from the above computations ∎

## Corollary 5:

$$\boxed{\begin{aligned} \det S_{\vec{p}} &= K(\vec{p}) \leftarrow \text{Gauss curvature} \\ \tfrac{1}{2}\operatorname{tr} S_{\vec{p}} &= H(\vec{p}) \leftarrow \text{Mean curvature} \end{aligned}}$$