B. Local Coordinates, Curvature, and Area

$\Sigma \subset \mathbb{R}^3$ a surface

$\vec{f} : V \longrightarrow \Sigma$ a coordinate chart, $V \subset \mathbb{R}^2$

for any $\vec{p} \in \operatorname{Im} \vec{f}$ we saw in section III that

$$T_{\vec{p}}\Sigma = \operatorname{Im} D\vec{f}_{\vec{q}} \qquad \text{where } \vec{q} \in V \text{ s.t. } \vec{f}(\vec{q}) = \vec{p}$$

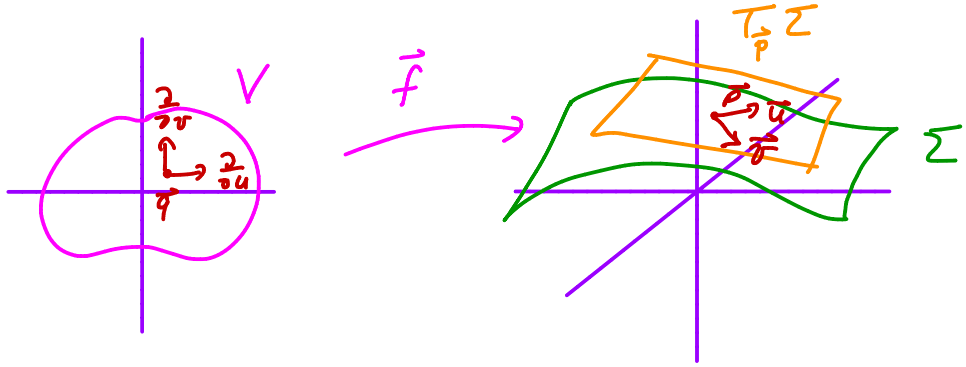
If we use coordinates (u, v) on $V \subset \mathbb{R}^2$ then $\operatorname{Im} D\vec{f}_{\vec{q}}$ is spanned by

$$\frac{\partial \vec{f}}{\partial u}(\vec{q}) \quad \text{and} \quad \frac{\partial \vec{f}}{\partial v}(\vec{q})$$

we denote these by

$$\vec{u}(\vec{p}) \quad \text{and} \quad \vec{v}(\vec{p})$$

recall we denote $\begin{bmatrix}1\\0\end{bmatrix}$ by $\frac{\partial}{\partial u}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ by $\frac{\partial}{\partial v}$



The 1^{st} fundamental form of $\Sigma \subset \mathbb{R}^3$ on $T_{\vec{p}}\Sigma$ is

$$I(\vec{p}) = T_{\vec{p}}\Sigma \times T_{\vec{p}}\Sigma \longrightarrow \mathbb{R}$$

$$(\vec{w}_1, \vec{w}_2) \longmapsto \vec{w}_1 \cdot \vec{w}_2$$

vectors in $\mathbb{R}^3$ dot product in $\mathbb{R}^3$

so $I(\vec{p})$ is an inner product on $T_{\vec{p}}\Sigma$

if we set
$$a(\vec{p}) = \vec{u}(\vec{p}) \cdot \vec{u}(\vec{p})$$
$$b(\vec{p}) = \vec{u}(\vec{p}) \cdot \vec{v}(\vec{p})$$
$$c(\vec{p}) = \vec{v}(\vec{p}) \cdot \vec{v}(\vec{p})$$

then $I(\vec{p})$ can be represented by the matrix

$$\begin{pmatrix} a(\vec{p}) & b(\vec{p}) \\ b(\vec{p}) & c(\vec{p}) \end{pmatrix}$$

to see this note any vectors $\vec{x}, \vec{y} \in T_{\vec{p}}\Sigma$ can be written

$$\vec{x} = x_1\vec{u} + x_2\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in the basis $\vec{u}, \vec{v}$

$$\vec{y} = y_1\vec{u} + y_2\vec{v} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

now
$$\begin{aligned} I(\vec{p})(\vec{x}, \vec{y}) &= \vec{x} \cdot \vec{y} \\ &= (x_1\vec{u} + x_2\vec{v}) \cdot (y_1\vec{u} + y_2\vec{v}) \\ &= x_1y_1\,a + (x_1y_2 + x_2y_1)b + x_2y_2\,c \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

so in the basis $\vec{u}, \vec{v}$

$$I(\vec{p}) = \begin{pmatrix} a(\vec{p}) & b(\vec{p}) \\ b(\vec{p}) & c(\vec{p}) \end{pmatrix}$$

and this measures lengths of vectors and angles between vectors in $T_{\vec{p}}\Sigma$

now for $\vec{q}$ in V define

$$g(\vec{q}) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

by

$$\begin{aligned} g(\vec{q})(\vec{w}_1, \vec{w}_2) &= I(\vec{f}(\vec{q}))(D\vec{f}_{\vec{q}}(\vec{w}_1), D\vec{f}_{\vec{q}}(\vec{w}_2)) \\ &= [D\vec{f}_{\vec{q}}(\vec{w}_1)] \cdot [D\vec{f}_{\vec{q}}(\vec{w}_2)] \end{aligned}$$

so we can represent g in the basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ by

$$g(\vec{q}) = \begin{pmatrix} a(\vec{f}(\vec{q})) & b(\vec{f}(\vec{q})) \\ b(\vec{f}(\vec{q})) & c(\vec{f}(\vec{q})) \end{pmatrix}$$

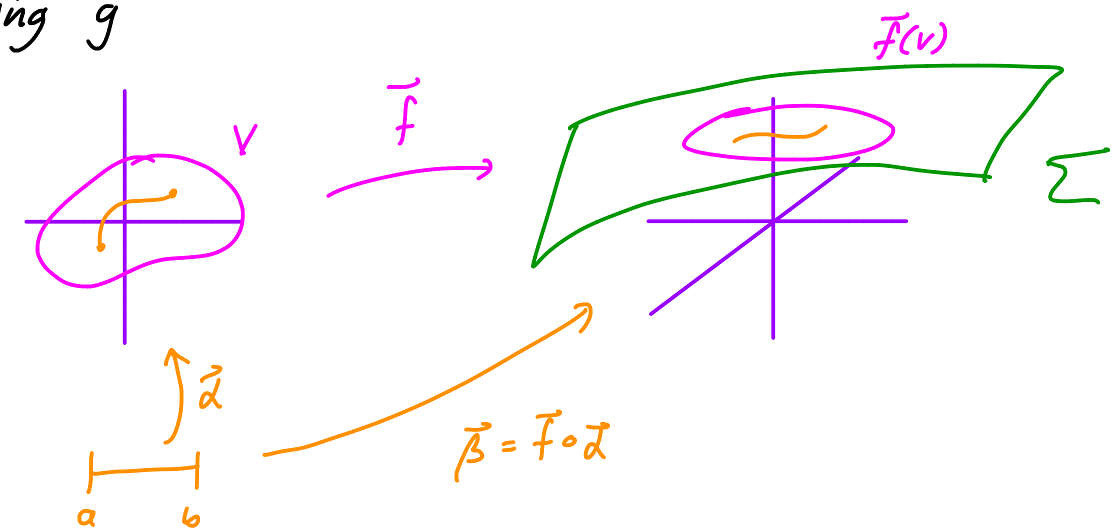
Idea is g represents I in local coordinates

$$\vec{f} : V \to \Sigma$$

we call g a Riemannian metric (but for $\Sigma \subset \mathbb{R}^3$ it is essentially equivalent to the 1^{st} fundamental form, more on this later)

many computations on Σ can be done in local coordinates using g

example:



$$\begin{aligned}
\text{length}(\vec{\beta}) &= \int_a^b \|\vec{\beta}'(t)\| \, dt \\
&= \int_a^b \sqrt{I(\vec{\beta}(t))(\vec{\beta}'(t), \vec{\beta}'(t))} \, dt \\
&= \int_a^b \sqrt{I(\vec{f}(\vec{\alpha}(t))(D\vec{f}_{\vec{\alpha}(t)}(\vec{\alpha}'(t)), D\vec{f}_{\vec{\alpha}(t)}(\vec{\alpha}'(t))} \, dt \\
&= \int_a^b \sqrt{g(\vec{\alpha}(t))(\vec{\alpha}'(t), \vec{\alpha}'(t))} \, dt \\
&= \int_a^b \|\vec{\alpha}'(t)\|_g \, dt
\end{aligned}$$

means length of vetor using g

Recall from calc III:

if a surface Σ in $\mathbb{R}^3$ is parameterized by

$$\vec{f} : V \longrightarrow \mathbb{R}^3 \qquad V \subset \underset{(u,v)}{\mathbb{R}^2}$$

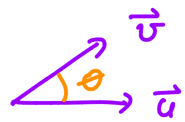
and $R = \vec{f}(V) \subset \Sigma$

then the area of R is

$$\boxed{\text{Area}(R) = \int_V \|\vec{u} \times \vec{v}\| \, du\, dv}$$

where $\vec{u} = \frac{\partial}{\partial u}\vec{f}$

$\vec{v} = \frac{\partial}{\partial v}\vec{f}$



note: $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$

$$= \|\vec{u}\|\|\vec{v}\| \sqrt{1 - \cos^2\theta}$$

$$= \|\vec{u}\| \|\vec{v}\| \sqrt{1 - \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)^2}$$

$$= \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2} = \sqrt{ac - b^2}$$

$$= \sqrt{\det I} = \sqrt{\det g}$$

so $\quad \text{Area}(R) = \int_V \sqrt{\det g}\, du\, dv$

example: $S^2 \subset \mathbb{R}^3$ be the unit sphere

$$\vec{f} : V \longrightarrow S^2 : (u,v) \longmapsto (u, v, \sqrt{1-u^2-v^2})$$

$$V = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$$

so $\vec{u} = \frac{\partial}{\partial u}\vec{f} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-u}{\sqrt{\ }} & \frac{-v}{\sqrt{\ }} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{-u}{\sqrt{\ }} \end{bmatrix}$

similarly $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ \frac{-v}{\sqrt{\ }} \end{bmatrix}$

so

$$a = \vec{u}\cdot\vec{u} = 1 + \frac{u^2}{1-u^2-v^2} = \frac{1-v^2}{1-u^2-v^2}$$

$$b = \vec{v}\cdot\vec{u} = \frac{uv}{1-u^2-v^2}$$

$$c = \vec{v}\cdot\vec{v} = \frac{1-u^2}{1-u^2-v^2}$$

and

$$g(u,v) = \frac{1}{1-u^2-v^2}\begin{pmatrix} 1-v^2 & uv \\ uv & 1-u^2 \end{pmatrix}$$

let U = upper hemisphere of S^3 so

$$\text{area}(U) = \int_V \sqrt{\det g}\; du\,dv$$

$$= \int_V \left(\frac{1}{(1-u^2-v^2)^2}\left((1-u^2)(1-v^2)-u^2v^2\right)\right)^{1/2} du\,dv$$

$$= \int_V \left(\frac{1}{(1-u^2-v^2)^2}(1-u^2-v^2)\right)^{1/2} du\,dv$$

$$= \int_V \frac{1}{\sqrt{1-u^2-v^2}}\, du\,dv$$

change to polar coord.s

$$= \int_V \frac{1}{\sqrt{1-r^2}}\, r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} \frac{r}{(1-r^2)}\, d\theta\, dr$$

$$= 2\pi \left(-\sqrt{1-r^2}\right)\Big|_0^1 = 2\pi$$

Recall to compute Gauss and mean curvature we can take the normal vector $\vec{N}$ to the surface and consider the shape operator

$$S_{\vec{p}}(\vec{v}) = -\vec{N}_{\vec{v}}(\vec{p})$$

then the Gauss curvature is $K(\vec{p}) = \det S_{\vec{p}}$ and the mean curvature is $H(\vec{p}) = \frac{1}{2}\,\mathrm{tr}\, S_p$ } by Cor. 5

now if $\vec{f}: V \to \Sigma$ gives local coordinates on Σ then $T_{\vec{p}}\Sigma$ is spanned by

$$\vec{u} = (D\vec{f}_{\vec{q}})\left(\frac{\partial}{\partial u}\right) \qquad \text{where } \vec{f}(\vec{q}) = \vec{p}$$

$$\vec{v} = (D\vec{f}_{\vec{q}})\left(\frac{\partial}{\partial v}\right)$$

and

$$\vec{N} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|}$$

in the basis $\vec{u}, \vec{v}$ for $T_{\vec{p}}\Sigma$ we can write $S_{\vec{p}}$ as

$$S_{\vec{p}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

just as we did above before lemma 3

we need to find a_{ij}

Set $A = S_{\vec{p}}(\vec{u}) \cdot \vec{u}$

$B = S_{\vec{p}}(\vec{u}) \cdot \vec{v}$

$C = S_{\vec{p}}(\vec{v}) \cdot \vec{v}$

Remark: The 2nd fundamental form of Σ at $\vec{p}$ is

$$\mathrm{II}(\vec{p}) : T_{\vec{p}}\Sigma \times T_{\vec{p}}\Sigma \to \mathbb{R}$$

$$(\vec{w}_1, \vec{w}_2) \longmapsto S_{\vec{p}}(\vec{w}_1) \cdot \vec{w}_2$$

so in the basis $\vec{u}, \vec{v}$ we can represent $\mathrm{II}(\vec{p})$ as

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

just like we did for the 1st fundamental form above

$S_{\vec{p}}$ and $\mathrm{II}(\vec{p})$ have the same information, so I only mention this since some books prefer II and some S

note:
$$A = S_{\vec{p}}(\vec{u})\cdot\vec{u} = (a_{11}\vec{u} + a_{21}\vec{v})\cdot\vec{u} = a_{11}a + a_{21}b$$
$$B = S_{\vec{p}}(\vec{u})\cdot\vec{v} = (a_{11}\vec{u} + a_{21}\vec{v})\cdot\vec{v} = a_{11}b + a_{21}c$$
$$S_{\vec{p}}(\vec{v})\cdot\vec{u} = (a_{12}\vec{u} + a_{22}\vec{v})\cdot\vec{u} = a_{12}a + a_{22}b$$
$$C = S_{\vec{p}}(\vec{v})\cdot\vec{v} = (a_{12}\vec{u} + a_{22}\vec{v})\cdot\vec{v} = a_{12}b + a_{22}c$$

this is equivalent to saying

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

or

$$\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1}$$

$$= \frac{1}{ac-b^2}\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

so we have proven the following

lemma 6:

with the above notation

$$K(\vec{p}) = \det S_{\vec{p}} = \frac{\det \mathrm{II}(\vec{p})}{\det \mathrm{I}(\vec{p})}$$

$$= \frac{AC - B^2}{ac - b^2}$$

and

$$H(\vec{p}) = \frac{1}{2} \operatorname{tr} S_{\vec{p}} = \frac{1}{2}\,\frac{Ac - 2bB + Ca}{ac - b^2}$$

Remark: In the proof of lemma 2 we saw that

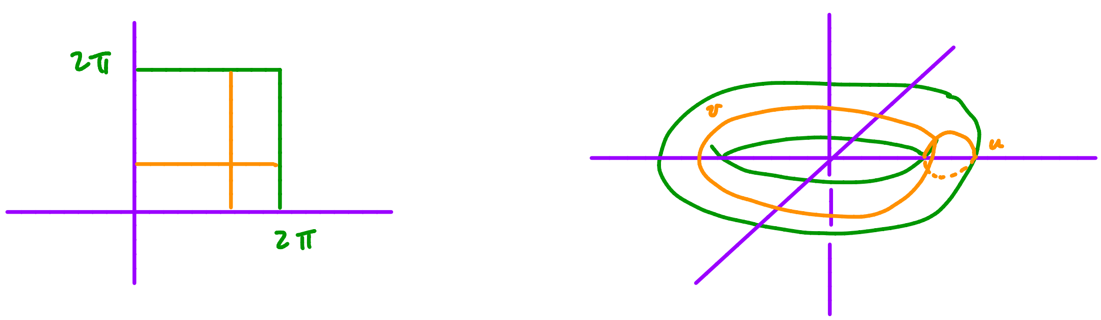
$$(\vec{N}_{\vec{u}})\cdot\vec{u} = \vec{N}\cdot\vec{f}_{\vec{u}\vec{u}}$$
$$(\vec{N}_{\vec{u}})\cdot\vec{v} = \vec{N}\cdot\vec{f}_{\vec{u}\vec{v}}$$
$$(\vec{N}_{\vec{v}})\cdot\vec{v} = \vec{N}\cdot\vec{f}_{\vec{v}\vec{v}}$$

where $\vec{f}_{\vec{u},\vec{v}}(\vec{p}) = \left[D\left[D\vec{f}_{\vec{q}}\left(\frac{\partial}{\partial u}\right)\right]_{\vec{q}}\left(\frac{\partial}{\partial v}\right)\right]$ for $\vec{q}$ st $\vec{f}(\vec{q}) = \vec{p}$

this can be very helpful when computing

example: $a > r > 0$ fixed constants

$$\vec{f}(u,v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u)$$



torus (donut)

$$\vec{u} = \vec{f}_{\vec{u}} = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u)$$
$$\vec{v} = \vec{f}_{\vec{v}} = (-(a + r\cos u)\sin v,\ (a + r\cos u)\cos v,\ 0)$$

so:
$$a = \vec{u}\cdot\vec{u} = r^2$$
$$b = \vec{u}\cdot\vec{v} = 0$$
$$c = (a + r\cos u)^2$$

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

and

$$\vec{N} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|} = \underbrace{\;\dots\;}_{\text{exercise}} = -(\cos u \cos v, \cos u \sin v, \sin u)$$

and

$$\vec{f}_{\vec{u}\vec{u}} = (-r\cos u \cos v, -r\cos u \sin v, -r\sin u)$$

$$\vec{f}_{\vec{v}\vec{u}} = (r\sin u \sin v, -r\sin u \cos v, 0)$$

$$\vec{f}_{\vec{v}\vec{v}} = (-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0)$$

so:

$$A = S(\vec{u})\cdot\vec{u} = \vec{N}\cdot\vec{f}_{\vec{u}\vec{u}} = r$$

$$B = S(\vec{u})\cdot\vec{v} = \vec{N}\cdot\vec{f}_{\vec{u}\vec{v}} = 0$$

$$C = S(\vec{v})\cdot\vec{v} = \vec{N}\cdot\vec{f}_{\vec{v}\vec{v}} = (a+r\cos u)\cos u$$

thus:

$$K(u,v) = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{AC-B^2}{ac-b^2} = \frac{r(a+r\cos u)\cos u}{r^2(a+r\cos u)^2}$$

$$= \frac{\cos u}{r(a+r\cos u)}$$