C. Some Implications of Curvature

note $S_{\vec p}$ is particularly simple if $\kappa_1 = \kappa_2 = C$

i.e $S_{\vec p}(\vec v) = C\vec v$

we call such a point $\vec p$ an umbilic point

Th^m 7:

> If every point of $\Sigma \subset \mathbb{R}^3$ is umbilic, then
>
> Σ is a part of a plane if $K = 0$ or
>
> Σ is a part of a sphere of radius $\frac{1}{\sqrt{K}}$ if $K > 0$
>
> (need Σ path connected for this)

note: the Gauss curvature at an umbilic point is ≥ 0

Proof: let $\vec f : \underset{(u,v)}{V} \longmapsto \Sigma \subset \mathbb{R}^3$ be a coord. chart

set
$$\left.\begin{aligned} \vec u &= D\vec f_{(u,v)}\left(\tfrac{\partial}{\partial u}\right) \\ \vec v &= D\vec f_{(u,v)}\left(\tfrac{\partial}{\partial v}\right) \end{aligned}\right\} \text{ span } T_{\vec f(u,v)}\Sigma$$

since all points are umbilic we have

K might depend on $\vec p$

$$\kappa \vec u = S_{\vec p}(\vec u) = -\vec N_{\vec u}$$
$$\kappa \vec v = S_{\vec p}(\vec v) = -\vec N_{\vec v}$$

In the proof of lemma 2.3 we saw

$$(\vec N_{\vec v})_{\vec u} = (\vec N_{\vec u})_{\vec v}$$

we actually did this for $\vec f$ but same computation works for $\vec N$ too

so

$$(\kappa \vec u)_{\vec v} = (\kappa \vec v)_{\vec u}$$

$$\kappa_{\vec v}\vec u + \kappa \vec u_{\vec v} = \kappa_{\vec u}\vec v + \kappa \vec v_{\vec u}$$

but recall $\quad \vec{u}_{\vec{v}} = \big(\underbrace{D\vec{f}\,(\tfrac{\partial}{\partial u})}_{\vec{u}}\big)_{\vec{v}} = D\left[D\vec{f}_{\vec{f}(u,v)}(\tfrac{\partial}{\partial u})\right]_{\vec{f}(u,v)}(\tfrac{\partial}{\partial v})$

$$\overset{\text{lemma 2.3 computation}}{=} D\left[D\vec{f}_{\vec{f}(u,v)}(\tfrac{\partial}{\partial v})\right]_{\vec{f}(u,v)}(\tfrac{\partial}{\partial u})$$

$$= \vec{v}_{\vec{u}}$$

so $\quad K_{\vec{v}}\,\vec{u} = K_{\vec{u}}\,\vec{v}$

and since $\vec{u}$ and $\vec{v}$ are linearly independent we see

$$K_{\vec{v}} = K_{\vec{u}} = 0$$

so K is constant on coordinate charts

exercise: Show K constant on all of Σ

Hint: $\vec{p}, \vec{q} \in \Sigma$ take path $\vec{\alpha}$ from $\vec{p}$ to $\vec{q}$
cover $\operatorname{im}(\vec{\alpha})$ by charts and consider
overlap of charts

Case 1: $K = 0$, then Σ part of a plane

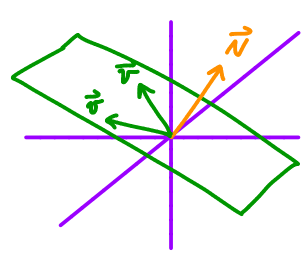
Proof: $K = 0 \Rightarrow S_{\vec{p}}(\vec{v}) = 0 \quad \forall \vec{v}$ and $\vec{p}$

so $\quad \vec{N}_{\vec{v}}(\vec{p}) = -S_{\vec{p}}(\vec{v}) = 0 \quad \forall \vec{v}$ and $\vec{p}$

Intuitively, $\vec{N}$ not changing so Σ must be in a
plane perpendicular to $\vec{N}$

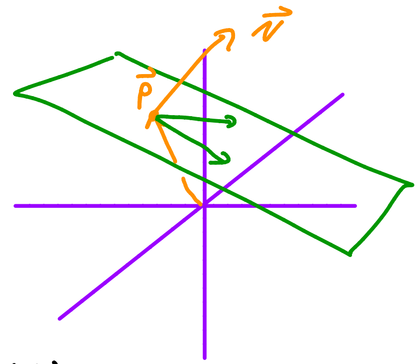
Rigorously: recall the equation for a plane perp. to $\vec{N}$ is

$$\vec{N} \cdot \vec{v} = 0$$



and the plane perp to $\vec{N}$ but through $\vec{p}$ is the set of all $\vec{v}$ satisfying

$$\vec{N}\cdot(\vec{v}-\vec{p}) = 0$$



for a fixed $\vec{p}\in\Sigma$ let $\vec{N} = \vec{N}(\vec{p})$

now for any $\vec{q}\in\Sigma$, let $\vec{\alpha}:[0,1]\to\Sigma$ be an arc with $\vec{\alpha}(0)=\vec{p}$ and $\vec{\alpha}(1) = q$

Set $f(t) = (\vec{\alpha}(t)-\vec{p})\cdot\vec{N}(\vec{\alpha}(t))$

$$f'(t) = \underbrace{\vec{\alpha}'(t)\cdot\vec{N}(\vec{\alpha}(t))}_{=0 \text{ since } \vec{\alpha}'(t)\in T_{\vec{\alpha}(t)}\Sigma} + (\vec{\alpha}(t)-\vec{p})\cdot \underbrace{D\vec{N}_{\vec{\alpha}(t)}(\vec{\alpha}'(t))}_{-S_{\vec{\alpha}(t)}(\vec{\alpha}'(t))}$$

$$= -(\vec{\alpha}(t)-\vec{p})\cdot S_{\vec{\alpha}(t)}(\vec{\alpha}'(t))$$

$$= 0$$

so $f(t)$ is constant!

$$f(0) = (\vec{p}-\vec{p})\cdot\vec{N}(\vec{p}) = 0$$

$\therefore$ $f(1) = 0$

$$\begin{aligned} f(1) &= 0 \\ &\parallel \\ (\vec{q}-\vec{p})&\cdot\vec{N}(\vec{q}) \\ &\parallel \\ (\vec{p}-\vec{q})&\cdot\vec{N} \end{aligned}$$

but $\frac{d}{dt}\vec{N}(\vec{\alpha}(t)) = D\vec{N}_{\vec{\alpha}(t)}(\vec{\alpha}'(t))$
$= -S_{\vec{\alpha}(t)}(\vec{\alpha}'(t))$
$= 0$
so $\vec{N}(\vec{q}) = \vec{N}(\vec{p}) = \vec{N}$

so for any point $\vec{q}\in\Sigma$ is in the plane

$$(\vec{v}-\vec{p})\cdot\vec{N} = 0 \checkmark$$

Case 2: $\kappa \neq 0$

Proof: let $\vec{f} : V \to \Sigma$ be a coord chart

Consider: $\vec{F}(u,v) = \vec{f}(u,v) + \frac{1}{\kappa} \vec{N}(\vec{f}(u,v))$

note: $\frac{\partial}{\partial u} \vec{F}(u,v) = \frac{\partial}{\partial u} \vec{f}(u,v) + \frac{1}{\kappa} \left(D\vec{N}_{\vec{f}(u,v)} \right) \left(\frac{\partial}{\partial u} \vec{f}(u,v) \right)$

$$= \frac{\partial}{\partial u} \vec{f}(u,v) - \frac{1}{\kappa} S_{\vec{f}(u,v)} \left(\frac{\partial}{\partial u} \vec{f}(u,v) \right)$$

$$= \frac{\partial}{\partial u} \vec{f} - \frac{\kappa}{\kappa} \frac{\partial}{\partial u} \vec{f} = 0$$

similarly $\frac{\partial}{\partial v} \vec{F} = 0$

so $\vec{F}$ is constant, say $\vec{F}(u,v) = \vec{c}$

that is, we have $\vec{f}(u,v) + \frac{1}{\kappa} \vec{N}(\vec{f}(u,v)) = \vec{c}$

so $\| \vec{f}(u,v) - \vec{c} \| = \| -\frac{1}{\kappa} \vec{N}(\vec{f}(u,v)) \| = \frac{1}{\kappa}$

that is any point in the image of $\vec{f}$ is on the sphere of radius $\frac{1}{\kappa}$ about $\vec{c}$

the Gauss curvature is $K = \kappa \kappa$

so the radius is $\frac{1}{\sqrt{K}}$

exercise: See other coord charts for Σ are on the same sphere

Thm 8:

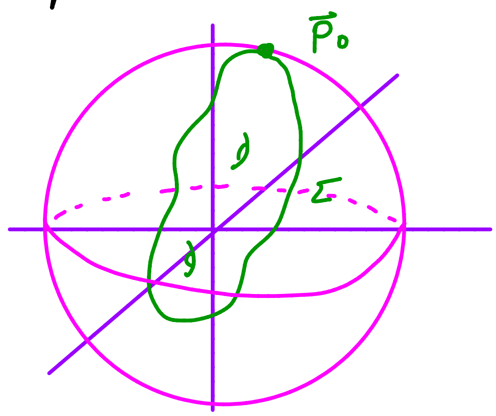
> If $\Sigma \subseteq \mathbb{R}^3$ is a compact surface, then there is some point $\vec{p} \in \Sigma$ such that $K(\vec{p}) > 0$

Proof: consider $f : \Sigma \to \mathbb{R} : \vec{p} \longmapsto \|\vec{p}\|^2 = \vec{p} \cdot \vec{p}$

f is continuous

a result from calculus implies that any continuous function on a compact set has a maximum (and min.)

so $\exists \, \vec{p}_0 \in \Sigma$ such that $f(\vec{p}) \leq f(\vec{p}_0) \quad \forall \vec{p} \in \Sigma$



so $\vec{p}_0$ is point on Σ farthest from origin

Idea: at $\vec{p}_0$, Σ curves more (in all directions) than the sphere of radius $\|\vec{p}_0\|$ about the origin

so all principal curvatures are larger than those of the sphere $\left(= \frac{1}{\|\vec{p}_0\|}\right)$

so we expect $K(p_0) \geq \frac{1}{\|\vec{p}_0\|^2} > 0$

To make this rigorous: compute the curvature of Σ at $\vec{p}_0$ in direction $\vec{u} \in U_{\vec{p}} \Sigma$

recall, there is a curve

$$\vec{\alpha} : [-\varepsilon, \varepsilon] \to \Sigma$$

such that
$$\vec{\alpha}(0) = \vec{p}_0$$
$$\vec{\alpha}'(0) = \vec{u}$$
$$\|\vec{\alpha}'(t)\| = 1 \quad \forall t$$

from lemma 1 we know

$$K_{\vec{p}_0}(\vec{u}) = \vec{\alpha}''(0) \cdot \vec{N}(\vec{p}_0)$$ normal to Σ

now lets find $\vec{N}$

for any tangent vector $\vec{v} \in T_{\vec{p}_0}\Sigma$ let

$\vec{\beta}$ be a path in Σ s.t.

$$\vec{\beta}(0) = \vec{p}_0$$

$$\vec{\beta}'(0) = \vec{v}$$

<u>note</u>: $f \circ \vec{\beta}(t) = \vec{\beta}(t) \cdot \vec{\beta}(t)$

so $\frac{d}{dt} f \circ \vec{\beta}(t)\Big|_{t=0} = \vec{\beta}'(0) \cdot \vec{\beta}(0) + \vec{\beta}(0) \cdot \vec{\beta}'(0)$

$$= 2\vec{p}_0 \cdot \vec{v}$$

but $f \circ \vec{\beta}$ has a maximum at 0 (since f on Σ does)

so $2\vec{p}_0 \cdot \vec{v} = \frac{d}{dt} f \circ \vec{\beta}\Big|_{t=0} = 0$

i.e. $\vec{v}$ is perpendicular to $\vec{p}_0$ $\forall \vec{v} \in T_{\vec{p}_0}\Sigma$

thus $\vec{N} = \frac{\vec{p}_0}{\|\vec{p}_0\|}$ is (or $-$ this, but sign wont matter)

now consider $f \circ \vec{\alpha} = \vec{\alpha} \cdot \vec{\alpha}$

since $t=0$ is a maximum of $f \circ \vec{\alpha}$ we see

$$\frac{d}{dt} f \circ \vec{\alpha}\Big|_{t=0} = 0$$

and

$$\frac{d^2}{dt^2} f \circ \vec{\alpha}\Big|_{t=0} \le 0$$

$$0 \ge \frac{d}{dt}\left(\frac{d}{dt} \vec{\alpha} \cdot \vec{\alpha}\right)\Big|_0 = \frac{d}{dt}\left(2\vec{\alpha}' \cdot \vec{\alpha}\right)\Big|_0$$

$$= 2\left(\vec{\alpha}'' \cdot \vec{\alpha} + \vec{\alpha}' \cdot \vec{\alpha}'\right)\Big|_0$$

$$= 2\left(\vec{\alpha}''(0) \cdot \vec{p}_0 + 1\right)$$

so $\vec{p}_0 \cdot \vec{\alpha}''(0) \le -1$

and
$$\kappa_{\vec{p}_0}(\vec{u}) = \vec{\alpha}''(0) \cdot \vec{N} = \vec{\alpha}''(0) \cdot \frac{\vec{p}_0}{\|\vec{p}_0\|}$$
$$\le -\frac{1}{\|\vec{p}_0\|} < 0$$

for any $\vec{u} \in U_{\vec{p}_0}\Sigma$

so both principal curvatures are $\le \frac{-1}{\|\vec{p}_0\|}$

$$\therefore K(p_0) = \kappa_1 \kappa_2 \ge \frac{1}{\|\vec{p}_0\|^2} > 0$$