V Intrinsic Geometry of Surfaces

A What is intrinsic geometry

recall geometry is the study of lengths and angles

so far we have had our surfaces in $\mathbb{R}^3$ and measured lengths and angles in a surface using $\mathbb{R}^3$ (and its dot product)

but if we lived in the surface Σ and did not know it was in $\mathbb{R}^3$ could we still "do geometry"?

the answer is yes!

recall from IV.B we saw that to measure lengths of vectors in a coordinate chart we can use the 1st fundamental form (a.k.a. Riemannian metric)

$$\underset{(u,v)}{\vec{f}: V} \to \Sigma \qquad \text{a coordinate chart}$$

then there is a matrix

$$g(u,v) = \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix}$$

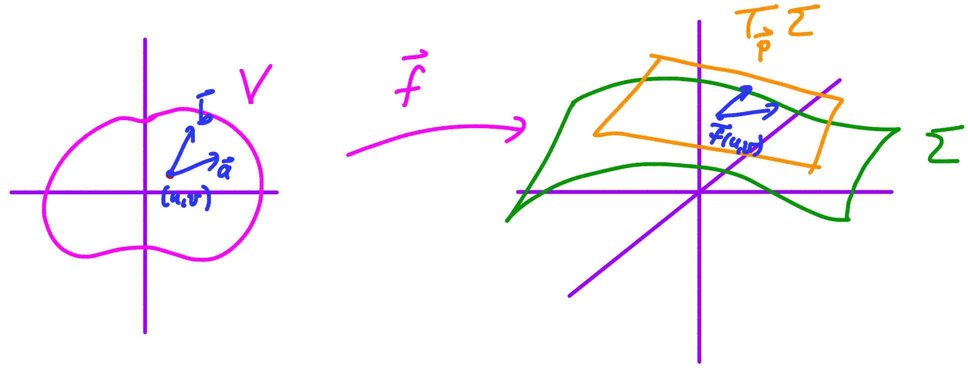
s.t. if $\vec{a}, \vec{b}$ are vectors in $\mathbb{R}^2$, then $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in the basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ and

$$(a_1 \;\; a_2) \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \vec{a}^{\,T} g(u,v) \, \vec{b}$$

$$= \left[D\vec{f}_{(u,v)}(\vec{a}) \right] \cdot \left[D\vec{f}_{(u,v)}(\vec{b}) \right]$$

by defn



so we can compute "dot products" (and hence lengths and angles) in $T_{\vec{p}}\Sigma$ by using the 1^{st} fundamental form in the coordinate chart $V \subset \mathbb{R}^2$ (i.e. Riemannian metric)

we also saw area can be computed in a coordinate chart just using g

so the 1^{st} fundamental form really encodes the "geometry" as seen "inside" the surface and recovers lengths (of vectors and curves), angles, and area

Can we see anything else?

probably not K_p normal curvature
or H mean curvature
why not?

but surprisingly

Th^{m} 1 [Theorema Egregium, Gauss]:

let (g_{ij}) be the 1^{st} fundamental form for a surface Σ (in local coordinates $\vec{f}: V \to \Sigma$)

K only depends on (g_{ij})
Gauss Curvature

in fact if $g_{12}=0$ (and hence $g_{21}=0$) then

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left\{\left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right)_v + \left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)_u\right\}$$

So Gauss curvature only depends on (g_{ij}) and not on 2^{nd} fundamental form!

Remark: We can always find coords where $g_{12}=0$ but in general we have

$$\boxed{K = \frac{1}{(\det g)^2}\left[\det\begin{pmatrix} g_{11} & g_{12} & (g_{12})_v - \frac{1}{2}(g_{22})_u \\ g_{21} & g_{22} & \frac{1}{2}(g_{22})_v \\ \frac{1}{2}(g_{11})_u & (g_{12})_v - \frac{1}{2}(g_{11})_v & -\frac{1}{2}(g_{11})_{vv} + (g_{12})_{uv} - \frac{1}{2}(g_{22})_{uu} \end{pmatrix} - \det\begin{pmatrix} g_{11} & g_{12} & \frac{1}{2}(g_{11})_v \\ g_{21} & g_{22} & \frac{1}{2}(g_{22})_u \\ \frac{1}{2}(g_{11})_v & (\frac{1}{2}g_{22})_u & 0 \end{pmatrix}\right]}$$

Proof: let $\vec{f}: V \to \Sigma$ be local coords on $\Sigma \subset \mathbb{R}^3$

use coords (u_1, u_2) on $V \subset \mathbb{R}^2$

let $\vec{f}_{u_i} = (D\vec{f}_{(u_1,u_2)})(\frac{\partial}{\partial u_i})$

and $\vec{f}_{u_i u_j} = D\left((D\vec{f}_{(u_1,u_2)})(\frac{\partial}{\partial u_i})\right)_{(u_1,u_2)}(\frac{\partial}{\partial u_j})$

we know $\vec{f}_{u_1}(u_1,u_2)$, $\vec{f}_{u_2}(u_1,u_2)$ span $T_{\vec{f}(u_1,u_2)}\Sigma$

so $\vec{f}_{u_1}(u_1,u_2)$, $\vec{f}_{u_2}(u_1,u_2)$, $\vec{N}(u_1,u_2)$ span $\mathbb{R}^3$ at $\vec{f}(u_1,u_2)$

Recall $K = \det II / \det I$ by lemma III.6 and $\det II$ can be computed by using $\vec{f}_{u_i u_j}$

we want to compute $\vec{f}_{u_i u_j}$ and these are vectors in $\mathbb{R}^3$ so they are linear combinations of $\vec{f}_{u_1}$, $\vec{f}_{u_2}$, $\vec{N}$

$$(*)\begin{cases} \vec{f}_{u_1u_1} = \Gamma_{11}^{1}\vec{f}_{u_1} + \Gamma_{11}^{2}\vec{f}_{u_2} + A\vec{N} \\ \vec{f}_{u_1u_2} = \vec{f}_{u_2u_1} = \Gamma_{12}^{1}\vec{f}_{u_1} + \Gamma_{12}^{2}\vec{f}_{u_2} + B\vec{N} \\ \vec{f}_{u_2u_2} = \Gamma_{22}^{1}\vec{f}_{u_1} + \Gamma_{22}^{2}\vec{f}_{u_2} + C\vec{N} \end{cases}$$

$\boxed{\Gamma_{jh}^{i} \text{ are called Christoffel symbols}}$

<u>note:</u> $\frac{d}{du_1} g_{11} = \frac{d}{du_1}(\vec{f}_{u_1}\cdot\vec{f}_{u_1}) = 2\vec{f}_{u_1u_1}\cdot\vec{f}_{u_1}$

$\frac{d}{du_1} g_{12} = \frac{d}{du_1}(\vec{f}_{u_1}\cdot\vec{f}_{u_2}) = \vec{f}_{u_1u_1}\cdot\vec{f}_{u_2} + \vec{f}_{u_1}\cdot\vec{f}_{u_2u_1}$

$\frac{d}{du_2} g_{11} = \frac{d}{du_2}(\vec{f}_{u_1}\cdot\vec{f}_{u_1}) = 2\vec{f}_{u_1u_2}\cdot\vec{f}_{u_1}$

you can take other 3 derivatives ($\frac{d}{du_1} g_{22}, \ldots$)

<u>note:</u> $\vec{f}_{u_1u_1}\cdot\vec{f}_{u_1} = \frac{1}{2}(g_{11})_{u_1} = \frac{1}{2}\left((g_{11})_{u_1} + (g_{11})_{u_1} - (g_{11})_{u_1}\right)$

$\vec{f}_{u_1u_1}\cdot\vec{f}_{u_2} = (g_{12})_{u_1} - \frac{1}{2}(g_{11})_{u_2}$

$= \frac{1}{2}\left((g_{12})_{u_1} + (g_{21})_{u_1} - (g_{11})_{u_2}\right)$

<u>exercise:</u> In general

$$\boxed{\vec{f}_{u_iu_j}\cdot\vec{f}_{u_k} = \frac{1}{2}\left((g_{jk})_{u_i} + (g_{ki})_{u_j} - (g_{ij})_{u_k}\right)}$$

<u>note:</u> $\begin{pmatrix} \vec{f}_{u_1u_1}\cdot\vec{f}_{u_1} \\ \vec{f}_{u_1u_1}\cdot\vec{f}_{u_2} \end{pmatrix} = \begin{pmatrix} g_{11}\Gamma_{11}^{1} + g_{12}\Gamma_{11}^{2} \\ g_{21}\Gamma_{11}^{1} + g_{22}\Gamma_{11}^{2} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix}$

so $\begin{pmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}\begin{pmatrix} \vec{f}_{u_1u_1}\cdot\vec{f}_{u_1} \\ \vec{f}_{u_1u_1}\cdot\vec{f}_{u_2} \end{pmatrix}$

where $(g^{ij}) = (g_{ij})^{-1}$

e.g. $\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \dfrac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$

so $\Gamma^1_{11} = g^{11}\frac{1}{2}\left((g_{11})_{u_1} + (g_{11})_{u_1} - (g_{11})_{u_1}\right) + g^{12}\frac{1}{2}\left((g_{12})_{u_1} - (g_{21})_{u_1} - (g_{11})_{u_2}\right)$

exercise: by considering $(g_{ij})\begin{pmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{pmatrix}$ and $(g_{ij})\begin{pmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{pmatrix}$

prove the following lemma

lemma 2:

with the notation above

$$\Gamma^i_{jk} = \sum_{\ell=1}^{2} g^{i\ell}\,\frac{1}{2}\left((g_{k\ell})_{u_j} + (g_{\ell j})_{u_k} - (g_{jk})_{u_\ell}\right)$$

Remark: note this says the tangential component of $\vec{f}_{u_i u_j}$ is completely determined by the 1st fundamental form and its derivatives! (i.e. intrinsic to the surface)

recall we are trying to compute

$$K(u,v) = \frac{\det \mathrm{II}(u,v)}{\det \underbrace{\mathrm{I}(u,v)}_{g}}$$

and $\mathrm{II} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

where
$$A = S_{\vec{p}}(\vec{f}_{u_1})\cdot\vec{f}_{u_1} = \vec{N}\cdot\vec{f}_{u_1 u_1}$$
$$B = S_{\vec{p}}(\vec{f}_{u_1})\cdot\vec{f}_{u_2} = \vec{N}\cdot\vec{f}_{u_1 u_2}$$
$$C = S_{\vec{p}}(\vec{f}_{u_2})\cdot\vec{f}_{u_2} = \vec{N}\cdot\vec{f}_{u_2 u_2}$$

to get A, B, C we differentiate $\circledast$ above and use

$$\vec{N}_{u_1} = -S_{\vec{p}}(\vec{f}_{u_1}) = -(a\vec{f}_{u_1} + b\vec{f}_{u_2})$$
$$\vec{N}_{u_2} = -S_{\vec{p}}(\vec{f}_{u_2}) = -(c\vec{f}_{u_1} + d\vec{f}_{u_2})$$

from Section IV.B just before lemma 6

$$(\vec{f}_{u_1u_1})_{u_2} = (\Gamma_{11}^1)_{u_2}\vec{f}_{u_1} + \Gamma_{11}^1\vec{f}_{u_1u_2} + (\Gamma_{11}^2)_{u_2}\vec{f}_{u_2} + \Gamma_{11}^2\vec{f}_{u_2u_2} + A_{u_2}\vec{N} + A\vec{N}_{u_2}$$
$$= (\Gamma_{11}^1)_{u_2}\vec{f}_{u_1} + \Gamma_{11}^1(\Gamma_{12}^1\vec{f}_{u_1} + \Gamma_{12}^2\vec{f}_{u_2} + B\vec{N})$$
$$+ (\Gamma_{11}^2)_{u_2}\vec{f}_{u_2} + \Gamma_{11}^2(\Gamma_{22}^1\vec{f}_{u_1} + \Gamma_{22}^2\vec{f}_{u_2} + C\vec{N})$$
$$+ A_{u_2}\vec{N} - A(c\vec{f}_{u_1} + d\vec{f}_{u_2})$$
$$= \left[(\Gamma_{11}^1)_{u_2} + \Gamma_{11}^1\Gamma_{12}^1 + \Gamma_{11}^2\Gamma_{22}^1 - Ac\right]\vec{f}_{u_1}$$
$$+ \left[(\Gamma_{11}^2)_{u_2} + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - Ad\right]\vec{f}_{u_2}$$
$$+ \left[A_{u_2} + \Gamma_{11}^1 B + \Gamma_{11}^2 C\right]\vec{N}$$

similarly

$$(\vec{f}_{u_1u_2})_{u_1} = \left[(\Gamma_{12}^1)_{u_1} + \Gamma_{12}^1\Gamma_{11}^1 + \Gamma_{12}^2\Gamma_{12}^1 - Ba\right]\vec{f}_{u_1}$$
$$+ \left[(\Gamma_{12}^2)_{u_1} + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - Bb\right]\vec{f}_{u_2}$$
$$+ \left[B_{u_1} + A\Gamma_{12}^1 + B\Gamma_{12}^2\right]\vec{N}$$

since $\vec{f}_{u_1u_2u_1} = \vec{f}_{u_1u_1u_2}$ we see coefficients on $\vec{f}_{u_1}, \vec{f}_{u_2}, \vec{N}$ must be the same

$$(\Gamma_{11}^1)_{u_2} + \cancel{\Gamma_{11}^1\Gamma_{12}^1} + \Gamma_{11}^2\Gamma_{22}^1 - Ac = (\Gamma_{12}^1)_{u_1} + \cancel{\Gamma_{12}^1\Gamma_{11}^1} + \Gamma_{12}^2\Gamma_{12}^1 - Ba$$
$$(\Gamma_{11}^2)_{u_2} + \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 - Ad = (\Gamma_{12}^2)_{u_1} + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{12}^2 - Bb$$

the second equation can be written

$$(\star)\quad (\Gamma^2_{11})_{u_2} - (\Gamma^2_{12})_{u_1} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2 = Ad - Bb$$

but recall from Section IV.B

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{II}\,(\mathrm{I}^{-1}) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}\frac{1}{g_{11}g_{22}-g_{12}g_{21}}$$

$$= \frac{1}{g_{11}g_{22}-g_{12}^2}\begin{pmatrix} Ag_{22} - Bg_{21} & Bg_{11} - Ag_{12} \\ Bg_{22} - Cg_{12} & Cg_{11} - Bg_{12} \end{pmatrix}$$

so
$$(\star) = \frac{1}{\det g}\left(A\left(Cg_{11} - \cancel{Bg_{12}}\right) - B\left(Bg_{11} - \cancel{Ag_{12}}\right)\right)$$

$$= \frac{1}{\det g}\left(g_{11}\left(AC - B^2\right)\right) = g_{11}\frac{\det \mathrm{II}}{\det \mathrm{I}} = g_{11}K$$

that is the Gauss curvature is

$$\boxed{K = \frac{1}{g_{11}}\left[(\Gamma^2_{11})_{u_2} - (\Gamma^2_{12})_{u_1} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})^2\right]}$$

this is all in terms of (g_{ij}) and its derivatives

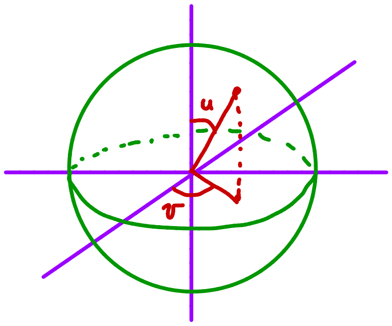
if $g_{12} = g_{21} = 0$ you can check this gives

$$K = \frac{-1}{2\sqrt{g_{11}g_{22}}}\left[\left(\frac{(g_{11})_{u_2}}{\sqrt{g_{11}g_{22}}}\right)_{u_2} + \left(\frac{(g_{22})_{u_1}}{\sqrt{g_{11}g_{22}}}\right)_{u_1}\right]$$

thus completing proof!

example: note the unit sphere has a coordinate chart

$$\vec{f}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$



$$\vec{f}_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$

$$\vec{f}_v = (-\sin u \sin v, \sin u \cos v, 0)$$

$$\vec{f}_{uu} = (-\sin u \cos v, -\sin u \sin v, -\cos u)$$

$$\vec{f}_{uv} = (-\cos u \sin v, \cos u \cos v, 0)$$

$$\vec{f}_{vv} = (-\sin u \cos v, -\sin u \sin v, 0)$$

note: $\vec{f}_{uu} = -\vec{f} = -\vec{N}$
so $\vec{f}_{uu}$ is a unit normal vector to S^2 at $\vec{f}(u,v)$

now $\vec{f}_{uu} = 0\vec{f}_u + 0\vec{f}_v - \vec{N}$

$$\vec{f}_{uv} = 0\vec{f}_u + \cot u\, \vec{f}_v + 0\vec{N}$$

$$\vec{f}_{vv} = -\sin u \cos u\, \vec{f}_u - \sin^2 u\, \vec{N}$$

so

$$\boxed{\begin{array}{ll} \Gamma^u_{uu} = 0 & \Gamma^v_{uu} = 0 \\ \Gamma^u_{uv} = 0 & \Gamma^v_{uv} = \cot u \\ \Gamma^u_{vv} = -\sin u \cos u & \Gamma^v_{vv} = 0 \end{array}}$$

also $\vec{f}_u \cdot \vec{f}_u = 1 \qquad \vec{f}_u \cdot \vec{f}_v = 0 \qquad \vec{f}_v \cdot \vec{f}_v = \sin^2 u$

so $$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$$

and $K = \frac{1}{1}\left[0 - \frac{d}{du}(\cot u) + 0 + 0 - 0 - \cot^2 u\right]$

$$= -\left(\frac{-\sin^2 u - \cos^2 u}{\sin^2 u}\right) - \frac{\cos^2 u}{\sin^2 u} = \frac{\sin^2 u}{\sin^2 u} = 1$$

alternatively

$$K = \frac{-1}{2\sqrt{g_{11}g_{22}}}\left(\left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right)_v + \left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)_u\right)$$

$$= \frac{-1}{2\sqrt{\sin^2 u}}\left(0 + \left(\frac{2\sin u\cos u}{\sqrt{\sin^2 u}}\right)_u\right)$$

$$= \frac{-1}{|\sin u|}\left(\frac{\sin u\cos u}{|\sin u|}\right)_u$$

$$= \begin{cases} \frac{\sin u}{|\sin u|} = 1 & \sin u > 0 \\ \frac{-\sin u}{|\sin u|} = 1 & \sin u < 0 \end{cases}$$

$$= 1$$

Remark: The normal curvatures and mean curvature depend on more than the 1st fundamental form

example: 1) Σ = xy-plane

$\vec{f}(u,v) = (u,v,0)$

$\vec{f}_u = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \vec{f}_v = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\vec{f}_u \cdot \vec{f}_u = 1 = \vec{f}_v \cdot \vec{f}_v$

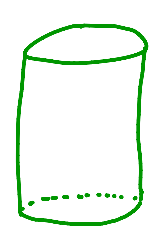
$\vec{f}_u \cdot \vec{f}_v = 0$ so $g = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

you can easily check all normal curvatures are 0

so all principal curvatures and mean curvature = 0

2) Σ = cylinder

$\vec{f}(u,v) = (\cos u, \sin u, v)$



$$\vec{f}_u = \begin{bmatrix} -\sin u \\ \cos u \\ 0 \end{bmatrix} \quad \vec{f}_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{f}_u \cdot \vec{f}_u = 1 = \vec{f}_v \cdot \vec{f}_v$$
$$\vec{f}_u \cdot \vec{f}_v = 0 \qquad \text{so } g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but you can check $\kappa_1 = 1$ and $\kappa_2 = 0$

so normal curvatures are different from plane

even though g is the same!

also $H = \frac{1}{2}$

we call two surfaces Σ, Σ' in $\mathbb{R}^3$ isometric if there is a smooth bijection $\vec{\phi} : \Sigma \to \Sigma'$

that preserves lengths and angles between vectors

i.e. $\left[D\vec{\phi}_{\vec{p}}(\vec{v})\right] \cdot \left[D\vec{\phi}_{\vec{p}}(\vec{w})\right] = \vec{v} \cdot \vec{w}$ if $\vec{v}, \vec{w} \in T_{\vec{p}}\Sigma$

If $\vec{f}_1 : V_1 \to \Sigma$ a local coord chart

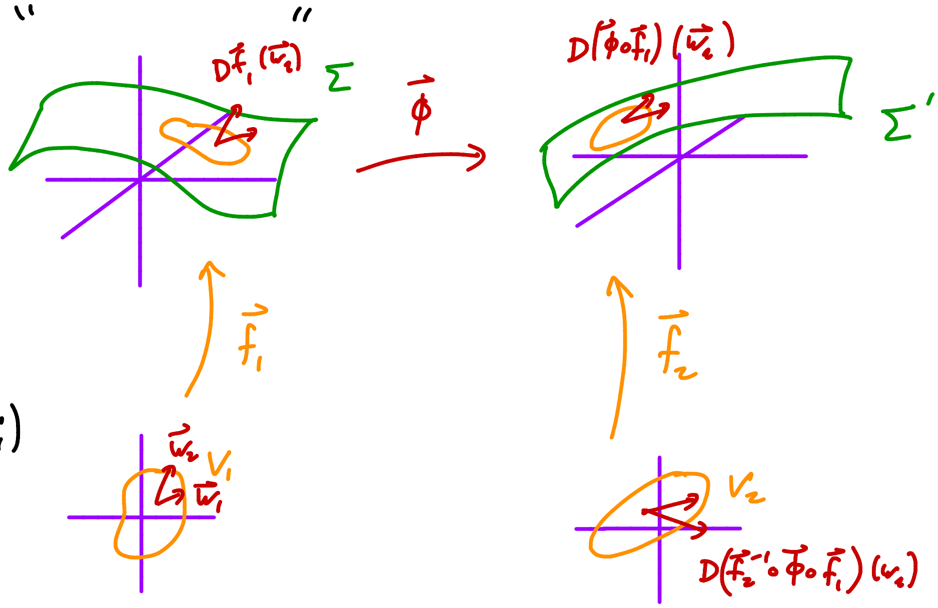
$\vec{f}_2 : V_2 \to \Sigma$ " "

such that

$\vec{\phi}(\vec{f}_1(V_1)) \subset \vec{f}_2(V_2)$

and (g_{ij}) is the 1st fundamental form for Σ in $(\vec{f}_1, V_1)$

(h_{ij}) is the 1st fundamental form for Σ in $(\vec{f}_2, V_2)$



then $\vec{\Phi}$ is an isometry (on $\vec{f}_1(V_1)$) if

$$\underbrace{\vec{w}_1^{\,t}\,(g_{ij}(\vec{p}))\,\vec{w}_2}_{\text{"dot product using } g\text{" in } V_1} = \Big[\underbrace{D(\vec{f}_2^{-1}\circ\vec{\phi}\circ\vec{f}_1)_{\vec{p}}(\vec{w}_1)}_{\vec{w}_1 \text{ moved to } V_2}\Big]^t \underbrace{\Big[h_{ij}(\vec{f}_2^{-1}\circ\vec{\phi}\circ\vec{f}_1(\vec{p}))\Big]}_{\text{"dot product using } h\text{" in } V_2}\Big[\underbrace{D(\vec{f}_2^{-1}\circ\vec{\phi}\circ\vec{f}_1)_{\vec{p}}(\vec{w}_2)}_{\vec{w}_2 \text{ moved to } V_2}\Big]$$

exercise: Prove this

lemma 3:

If $\phi: \Sigma \to \Sigma'$ is an isometry, then

$$\underbrace{K_{\Sigma'}(\vec{\Phi}(\vec{p}))}_{\text{Gauss Curvature at } \vec{\Phi}(\vec{p}) \text{ in } \Sigma'} = \underbrace{K_{\Sigma}(\vec{p})}_{\text{Gauss Curvature at } \vec{p} \text{ in } \Sigma}$$

Proof: from above if ϕ is an isometry, then 1st fundamental forms are the same, so by Thm 1, Gauss curvature same ∎

Corollary 4:

There is no map preserving lengths and angles from <u>any</u> neighborhood of a point on round S^2 to flat plane (Maps of the world can't be that good!)

A Riemannian metric on a domain V in $\mathbb{R}^2$ is a smooth map

$$g: V \to GL(2,\mathbb{R})$$

2x2 matricies with det $\neq 0$

such that $g_{(u,v)} = (g_{ij}(u,v))$ is symmetric for all $(u,v) \in V$

if $\vec{v}, \vec{w}$ are vectors in $\mathbb{R}^2$ (based at (u,v)), the define

$$g_{(u,v)}(\vec{v},\vec{w}) = \vec{v}^{\,t}(g_{(u,v)})\,\vec{w}$$

this is an inner product on vectors based at (u,v)

examples: 1) on $\mathbb{R}^2$ consider

$$g = \frac{4}{(1+u^2+v^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

(this is the 1^{st} fundamental form on S^2 in stereographic coord. chart)

2) on $D^2 = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$

$$g = \frac{4}{(1-(u^2+v^2))^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

(called the hyperbolic metric on $\mathbb{R}^2$)

Consider any surface Σ

(so $\Sigma \subset \mathbb{R}^n$, some n)

a Riemannian metric on Σ is

① a collection of local coordinate charts

$$\{\vec{f}_i : V_i \to \Sigma\} \quad \text{and}$$

② Riemannian metrics

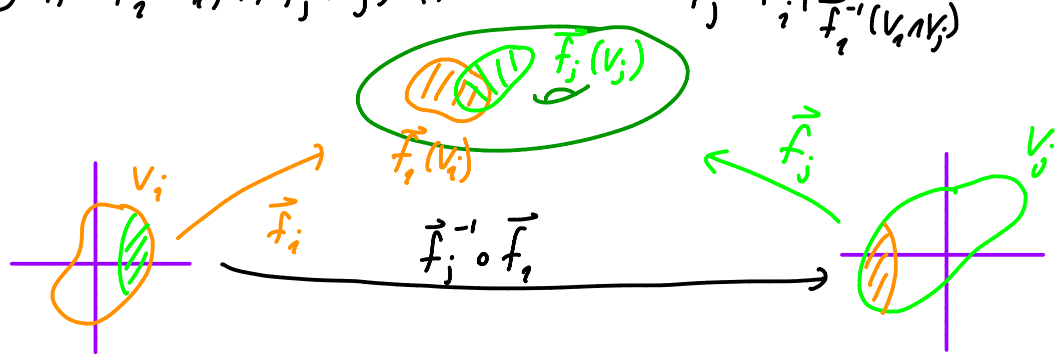
$$g_i : V_i \to GL(2,\mathbb{R})$$

such that

ⓐ the sets $\{\vec{f}_i(V_i)\}$ cover Σ

i.e. $\Sigma = \bigcup \vec{f}_i(V_i)$ and

ⓑ if $\vec{f}_i(V_i) \cap \vec{f}_j(V_j) \neq \emptyset$ then $\vec{f}_j^{-1} \circ \vec{f}_i \big|_{\vec{f}_i^{-1}(V_i \cap V_j)}$ is an isometry



note: this says if $\vec{v},\vec{w} \in T_{\vec{p}}\Sigma$ with $\vec{p} \in \vec{f}_i(V_i) \cap \vec{f}_j(V_j)$

then

$$(g_i)_{\vec{f}_i^{-1}(\vec{p})}\left((d\vec{f}_i^{-1})_{\vec{p}}(\vec{v}),(d\vec{f}_i^{-1})_{\vec{p}}(\vec{w})\right) = (g_j)_{\vec{f}_j^{-1}(\vec{p})}\left((d\vec{f}_j^{-1})_{\vec{p}}(\vec{v}),(d\vec{f}_j^{-1})_{\vec{p}}(\vec{w})\right)$$

that is we have a well-defined inner product of $\vec{v}$ and $\vec{w}$ which we denote $g_{\vec{p}}(\vec{v},\vec{w})$

i.e. we have a 1st fundamental form on Σ

so a Riemannian metric g on Σ allows us to talk about

- lengths of vectors $\vec{v} \in T_{\vec{p}}\Sigma$

$$\|\vec{v}\|_g = \sqrt{g_{\vec{p}}(\vec{v},\vec{v})}$$

- angles between vectors $\vec{v},\vec{w} \in T_{\vec{p}}\Sigma$

$$\theta = \cos^{-1}\frac{g_{\vec{p}}(\vec{v},\vec{w})}{\|\vec{v}\|_{\vec{g}}\,\|\vec{w}\|_{\vec{g}}}$$

- lengths of curves $\vec{\alpha}:[a,b]\to\Sigma$

$$\ell(\vec{\alpha}) = \int_a^b \|\vec{\alpha}'(t)\|\,dt$$

- area of regions $U \subset \Sigma$

$$A(U) = \int_U \sqrt{\det g}\,du\,dv$$

- Gauss curvature (using Thm 1)

(even though Σ not necessarily in $\mathbb{R}^3$)

lemma 5:

$$\boxed{\text{Any surface } \Sigma \text{ has a Riemannian metric}}$$

Proof: by definition $\Sigma \subset \mathbb{R}^n$ some n

take a collection of coordinate charts

$$\vec{f}_i : V_i \to \Sigma$$

that cover Σ

let $g_i : V_i \to GL(2,\mathbb{R})$ be defined by the dot product in $\mathbb{R}^n$

i.e. $(g_i)_{(u,v)}(\vec{v},\vec{w}) = \left(D(\vec{f}_i)_{(u,v)}(\vec{v})\right) \cdot \left(D(\vec{f}_i)_{(u,v)}(\vec{w})\right)$

<u>exercise</u>: show these g_i define a Riemannian metric

<u>example</u>: the flat torus in $\mathbb{R}^4$

$$\vec{f} : \mathbb{R}^2 \to T^2 \subset \mathbb{R}^4$$

$$(u,v) \mapsto (\cos u, \sin u, \cos v, \sin v)$$

you can check

$$\vec{f}_u \cdot \vec{f}_u = 1 = \vec{f}_v \cdot \vec{f}_v$$

$$\vec{f}_u \cdot \vec{f}_v = 0$$

so $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $K = 0$ on all of T^2

<u>Remark</u>: this Riemannian metric on T^2 cannot come from T^2 sitting in $\mathbb{R}^3$ because we showed any T^2 (compact surface) in $\mathbb{R}^3$ has a point with $K > 0$ (Thm IV.8)