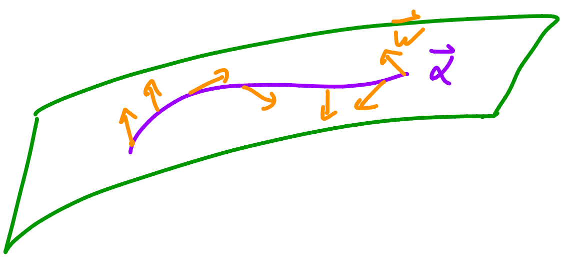
B. Covariant Derivatives and Parallel Transport

let $\Sigma \subset \mathbb{R}^3$ be a surface

$\vec{\alpha} : (-\varepsilon, \varepsilon) \to \Sigma$ a curve

$\vec{w} : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ a vector field along $\vec{\alpha}$

i.e. $\vec{w}(t) \in T_{\vec{\alpha}(t)}\Sigma$



$\frac{d\vec{w}}{dt}(0)$ is a vector in $\mathbb{R}^3$ at $\vec{\alpha}(0)$

we define the covariant derivative of $\vec{w}$ in the direction $\vec{v} = \vec{\alpha}'(0)$

at $\vec{p} = \vec{\alpha}(0)$ to be

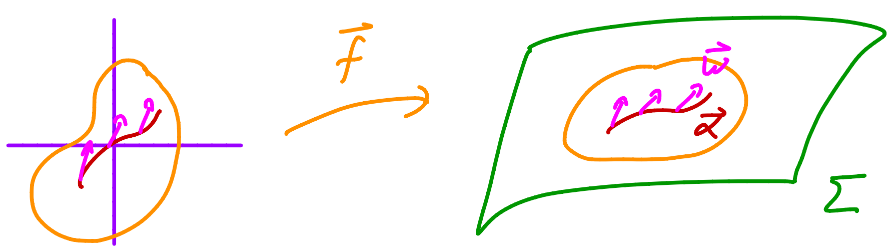
$$\nabla_{\vec{v}}\vec{w} = \text{the projection of } \frac{d\vec{w}}{dt}(0) \text{ to } T_{\vec{p}}\Sigma \text{ (along } \vec{N})$$
$$= \frac{d\vec{w}}{dt}(0) - \left(\frac{d\vec{w}}{dt}(0) \cdot \vec{N}\right)\vec{N}$$

also denoted $\frac{D}{dt}\vec{w}$

Remark: This is the part of $\frac{d\vec{w}}{dt}$ that you can "see" in Σ

in local coordinates:

$\vec{f} : V \to \Sigma$ coord chart



there is some curve $t \mapsto (u(t), v(t))$ in V

such that $\vec{f}(u(t), v(t)) = \vec{\alpha}(t)$

also $\vec{f}_u$ and $\vec{f}_v$ span $T\Sigma$ so there are numbers $a(t), b(t)$ such that

$$\vec{w}(t) = a(t)\,\vec{f}_u(u(t),v(t)) + b(t)\,\vec{f}_v(u(t),v(t))$$

Recall from Section A we have

$$\begin{aligned}
\vec{f}_{u_1u_1} &= \Gamma^1_{11}\vec{f}_{u_1} + \Gamma^2_{11}\vec{f}_{u_2} + A\vec{N} \\
\vec{f}_{u_1u_2} = \vec{f}_{u_2u_1} &= \Gamma^1_{12}\vec{f}_{u_1} + \Gamma^2_{12}\vec{f}_{u_2} + B\vec{N} \\
\vec{f}_{u_2u_2} &= \Gamma^1_{22}\vec{f}_{u_1} + \Gamma^2_{22}\vec{f}_{u_2} + C\vec{N}
\end{aligned}$$

and lemma 2 gives a formula for Γ^i_{jk} in terms of the Riemannian metric (ie. 1^{st} fundamental form)

now

$$\begin{aligned}
\frac{d\vec{w}}{dt} &= \frac{da}{dt}\vec{f}_u + a\frac{d}{dt}\vec{f}_u + \frac{db}{dt}\vec{f}_v + b\frac{d}{dt}\vec{f}_v \\
&= a'\vec{f}_u + a\left(\vec{f}_{uu}u'(t) + \vec{f}_{uv}v'(t)\right) + b'\vec{f}_v + b\left(\vec{f}_{vu}u'(t) + \vec{f}_{vv}v'(t)\right) \\
&= a'\vec{f}_u + a\left(\left(\Gamma^1_{11}\vec{f}_u + \Gamma^2_{11}\vec{f}_v + A\vec{N}\right)u' + \left(\Gamma^1_{12}\vec{f}_u + \Gamma^2_{12}\vec{f}_v + B\vec{N}\right)v'\right) \\
&\quad + b'\vec{f}_v + b\left(\left(\Gamma^1_{12}\vec{f}_u + \Gamma^2_{12}\vec{f}_v + B\vec{N}\right)u' + \left(\Gamma^1_{22}\vec{f}_u + \Gamma^2_{22}\vec{f}_v + C\vec{N}\right)v'\right)
\end{aligned}$$

thus

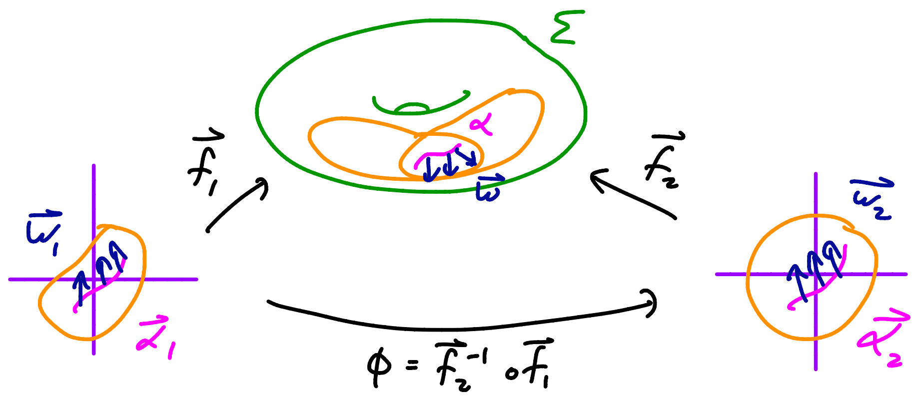
$$\boxed{\begin{aligned}
\nabla_{\vec{v}}\vec{w} = &\left[a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv'\right]\vec{f}_u \\
&+ \left[b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^1_{12}bu' + \Gamma^2_{22}bv'\right]\vec{f}_v
\end{aligned}} \quad (*)$$

we have proven

lemma 6: The covariant derivative is intrinsic defined by $(*)$ ie only depends on the 1^{st} fundamental form and hence can be defined for any surface with a Riemannian metric even if it is not in $\mathbb{R}^3$!

Remark: It is interesting to note that if you compute $\nabla_{\vec{\gamma}}\vec{w}$ in different coordinate charts you get the same thing but $\frac{d}{dt}\vec{w}$ is not the same

i.e.



$\vec{\alpha}$ in Σ is $\vec{f}_1 \circ \vec{\alpha}_1$ and $\vec{f}_2 \circ \vec{\alpha}_2$

and $D\vec{f}_1(\vec{w}_1) = \vec{w} = D\vec{f}_2(\vec{w}_2)$

then $D\phi\left(\frac{d\vec{w}_1}{dt}\right) \neq \frac{d\vec{w}_2}{dt}$

but $D\phi\left(\nabla_{\vec{\alpha}_1(0)} \vec{w}_1\right) = \nabla_{\vec{\alpha}_2(0)} \vec{w}_2$

exercise: Check this

Given a vectorfield $\vec{w}(t)$ along a curve $\vec{\alpha}(t)$ we say $\vec{w}$ is **parallel** if

$$\nabla_{\vec{\alpha}'(t)} \vec{w}(t) = 0 \quad \text{for all } t$$

example: in $\mathbb{R}^2$

$$\vec{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \text{ is parallel along } \vec{\alpha}(t)$$

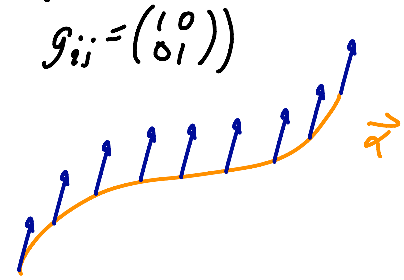
$$\Longleftrightarrow$$

$$w_1'(t) = w_2'(t) = 0 \quad \left(\text{since all } \Gamma^i_{jk} = 0 \text{ because } g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\Longleftrightarrow$$

w_1 and w_2 constant

so the lengths of $\vec{w}(t)$ don't change



Thm 7:

let Σ be a surface with a Riemanian metric g. If $\vec{v}, \vec{w}$ are parallel vector fields along

$$\vec{\alpha} : (a,b) \to \Sigma$$

then

1) $\|\vec{v}\|_g$ and $\|\vec{w}\|_g$ are constant
2) $\langle \vec{v}, \vec{w} \rangle_g$ is constant
3) angle between $\vec{v}$ and $\vec{w}$ constant

Proof: recall $\langle \vec{v}, \vec{w} \rangle_g$ is like a "dot product" so we expect a "product rule" for derevatives

exercise: show

$$\frac{d}{dt} \langle \vec{v}(t), \vec{w}(t) \rangle_g = \langle \nabla_{\vec{\alpha}'(t)} \vec{v}(t), \vec{w}(t) \rangle_g + \langle \vec{v}(t), \nabla_{\vec{\alpha}'(t)} \vec{w}(t) \rangle_g$$

given this we clearly have for parallel $\vec{v}$ and $\vec{w}$

$$\frac{d}{dt} \langle \vec{v}(t), \vec{w}(t) \rangle_g = \langle 0, \vec{w} \rangle_g + \langle \vec{v}, 0 \rangle_g = 0$$

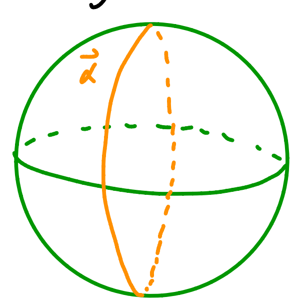
so $\langle \vec{v}(t), \vec{w}(t) \rangle_g$ is constant 1) & 3) follow ▦

example:

let $S^2 \subset \mathbb{R}^3$ be the unit sphere

let $\vec{\alpha}$ be a unit speed parameterization of a great circle

let $\vec{w}(t) = \vec{\alpha}'(t)$



note: $\frac{d\vec{w}}{dt} = \vec{\alpha}''(t)$

which is perpendicular to $T_{\vec{\alpha}(t)} S^2$

(recall for such a curve $\vec{\alpha}''(t)$ is $-\vec{\alpha}(t)$)

so $\nabla_{\vec{\alpha}'(t)}\vec{w}(t)=0$ and $\vec{w}$ is "parallel" along $\vec{\alpha}$ in S^2

<u>note</u>: $\vec{w}$ is <u>not</u> "parallel" in the usual sense in $\mathbb{R}^3$
but all the "twisting" of $\vec{w}$ is to keep it tangent to S^2

Th$^{\underline{m}}$8:

given $\alpha:[c,d]\to\Sigma$ a parameterization of a curve in Σ
a Riemannian metric g on Σ
(or if $\Sigma\subset\mathbb{R}^3$ its 1^{st} fundamental form)
$\vec{w}_0\in T_{\vec{\alpha}(c)}\Sigma$ a tangent vector

Then there is a <u>unique</u> parallel vector field $\vec{w}(t)$
along $\vec{\alpha}(t)$ with $\vec{w}(c)=\vec{w}_0$

<u>Proof</u>: let $\vec{f}:V\to\Sigma$ be a coordinate chart containing the curve $\vec{\alpha}$

<u>exercise</u>: think about what happens if $\vec{\alpha}$ not in a single chart

then as above there are functions $u(t), v(t)$ such that

$$\vec{\alpha}(t)=\vec{f}(u(t),v(t))$$

and

$$\vec{\alpha}'(t)=\vec{f}_u(u(t),v(t))\,u'(t)+\vec{f}_v(u(t),v(t))\,v'(t)$$

we are looking for a vector field

$$\vec{w}(t)=a(t)\vec{f}_u(t)+b(t)\vec{f}_v(t)$$

that is we are looking for functions $a(t), b(t)$ s.t.

$$\nabla_{\vec{\alpha}'(t)}\vec{w}(t)=0$$

now

by ⊛ this is the definition of covariant derivative

$$\nabla_{\vec{\alpha}'(t)}\vec{w}(t)=\left[a'+au'\Gamma_{11}^1+av'\Gamma_{12}^1+bu'\Gamma_{12}^1+bv'\Gamma_{22}^1\right]\vec{f}_u$$
$$+\left[b'+au'\Gamma_{11}^2+av'\Gamma_{12}^2+bu'\Gamma_{12}^1+bv'\Gamma_{22}^2\right]\vec{f}_v$$

$= \vec{0} \quad = 0\,\vec{f}_u + 0\,\vec{f}_v$

want

<u>note</u>: u, v are known function determined by $\vec{\alpha}$

Γ_{ij}^k " " determined by g

so we have two linear ordinary differential equations for a and b

$$a' + au'\Gamma_{11}^1 + av'\Gamma_{12}^1 + bu'\Gamma_{12}^1 + bv'\Gamma_{22}^1 = 0$$

$$b' + au'\Gamma_{11}^2 + av'\Gamma_{12}^2 + bu'\Gamma_{12}^1 + bv'\Gamma_{22}^2 = 0$$

we also have the initial condition that $a(c) = w_1$ and $b(c) = w_2$

where $\vec{w}_0 = w_1 \vec{f}_u + w_2 \vec{f}_v$

now the standard theory of ODE's says there is a unique sol$^{\underline{n}}$ $a(t)$ and $b(t)$ given these initial conditions

this defines $\vec{w}(t)$

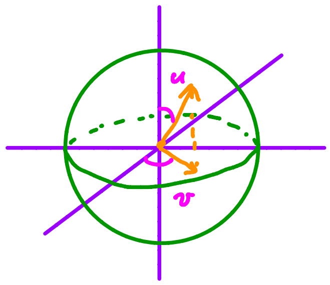
<u>definition:</u> with $\vec{\alpha}, g,$ and $\vec{w}_0$ as in the theorem

$\vec{w}(t) \in T_{\vec{\alpha}(t)}\Sigma$ is called the <u>parallel transport</u> of $\vec{w}_0$ along $\vec{\alpha}$

this is the Riemannian geometry analog of just "translating" vectors in $\mathbb{R}^n$

<u>note</u>: Th$^{\underline{m}}$ says parallel transport only depends on 1^{st} fundamental form (i.e. Riemannian metric) and the path $\vec{\alpha}$

example: consider the unit sphere with parameterization



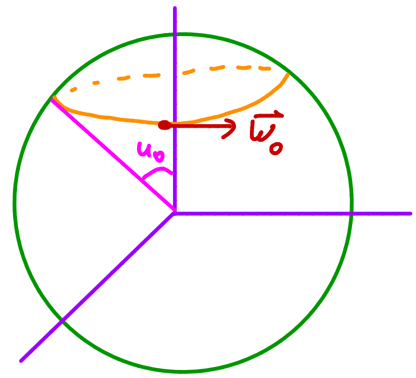
$$\vec{f}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

we computed $\vec{f}_u, \vec{f}_v$, and Γ_{ij}^k in section A

let the curve c be at "lattitude" $\pi/2 - u_0$

and $\vec{w}_0 = \vec{f}_v(u_0, 0)$

$= (\cos u_0, 0, \sin u_0)$



we want to compute the parallel transport of $\vec{w}_0$ along C

note C is parameterized by

$\vec{f}(u(t), v(t))$ where $u(t) = u_0$
$v(t) = t$, $t \in [0, 2\pi]$

we are now trying to find $a(t), b(t)$ s.t.

$$\vec{w}(t) = a(t)\vec{f}_u + b(t)\vec{f}_v$$

is parallel along $\vec{\alpha}$ and $\vec{w}(0) = \vec{w}_0$

from the proof a, b must satisfy

$$a' + \overset{=0}{a u' \Gamma_{11}^1} + \overset{=0}{a v' \Gamma_{12}^1} + \overset{=0}{b u' \Gamma_{12}^1} + b \overset{1}{v'} \overset{-\sin u(t) \cos u(t),\ u(t)=u_0}{\Gamma_{22}^1} = 0$$

① $\boxed{a'(t) - \sin u_0 \cos u_0 \, b(t) = 0}$

and

$$b' + a \overset{=0}{u'} \Gamma_{11}^2 + a \overset{=1}{v'} \overset{= \cot u_0}{\Gamma_{12}^2} + b \overset{=0}{u'} \Gamma_{12}^2 + b v' \overset{=0}{\Gamma_{22}^2} = 0$$

② $\boxed{b'(t) + \cot u_0 \, a(t) = 0}$

and we want $\vec{w}(0) = \vec{w}_0 = \vec{f}_v$

so $a(0) = 0$, $b(0) = 1$

to solve we first differentiate ② to get

$$b''(t) = -a'(t)\cot u_0 \overset{①}{=} -b(t)\cos u_0 \sin u_0 \cot u_0$$
$$= -b(t)\cos^2 u_0$$

we know from a Diff $Eq^{\underline{n}}$ course, the $sol^{\underline{n}}$ to any equation of the form

$$y''(t) = -k^2 y(t)$$

is $y(t) = c_1 \cos kt + c_2 \sin kt$

so we see

$$b(t) = c_1 \cos(\cos u_0 t) + c_2 \sin(\cos u_0 t)$$

since we want $b(0) = 1$ we see $c_2 = 0$ and $c_1 = 1$

so $b(t) = \cos((\cos u_0)t)$

now $a(t) = \int_0^t \sin u_0 \cos u_0 \, b(t)\, dt + c_3$

$$= \sin u_0 \cos u_0 \frac{1}{\cos u_0} \sin((\cos u_0)t) + c_3$$

since $a(0) = 0$ need $c_3 = 0$

so we see

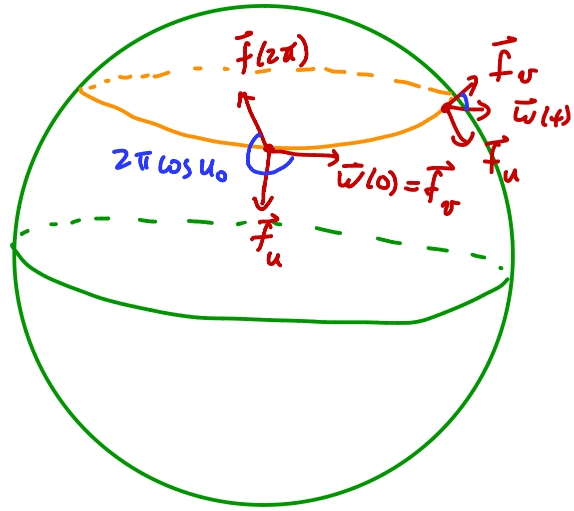
$$\boxed{\begin{aligned} a(t) &= \sin u_0 \sin((\cos u_0)t) \\ b(t) &= \cos((\cos u_0)t) \end{aligned}}$$

solves ① and ② and so the vector field

$$\vec{w}(t) = \sin u_0 (\sin(\cos u_0)t)\vec{f}_u + \cos((\cos u_0)t)\vec{f}_v$$

is parallel along $\vec{\alpha}(t)$

note: when you go along C, $\vec{w}$ rotates by



$$\theta(t) = \cos^{-1}\left(\frac{\vec{f}_v \cdot \vec{w}(t)}{\|\vec{f}_v\|\,\|\vec{w}(t)\|}\right)$$

$$= \cos^{-1}\left(\frac{\cos((\cos u_0)t)\ \sin^2 u_0}{\sin u_0 \sin u_0}\right)$$

$$= (\cos u_0)\,t$$

note: at the equator $\vec{w}(t) = \vec{f}_v(t)$ $\forall t$

ie $\vec{w}(t) = \alpha'(t)$

as we know since equator is a great circle

so <u>no</u> rotatoin

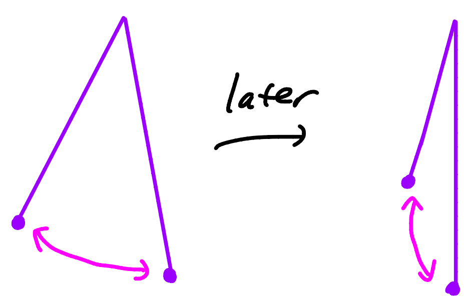
but as C gets closer to a pole $\vec{w}(t)$ rotates faster!

Great Application:

recall lattitude measured from equator

Foucault observed in 1851 that a pendulum at "lattitude" ϕ precesses with an angular speed of $T = 2\pi\sin\phi$ radians/day

that is the plane in which the pendulum swings rotates



we can understand this in terms of parallel transport

Idea: if pendulum is very long then its motion (almost) gives a tangent vector to the sphere (its motion is (almost) a path on sphere)

we think of the earth as not moving but we transport the pendulum around the lattitude circle so we get back to the start in 24 hours

since we are moving slowly and applying no "force" to the pendulum, we expect its swing to "not change" or be "parallel" as we move

thus above computation shows the plane in which the pendulum swings will rotate by $2\pi \cos u_0$ so the "speed" of rotation is

$$2\pi \cos u_0 = 2\pi \cos\left(\frac{\pi}{2} - \phi\right) = 2\pi \sin \phi \text{ radians/day}$$

C. Geodesics

we call a regular parameterized curve

$$\vec{\alpha} : [a,b] \to \Sigma$$

a geodesic if its tangent vector is parallel along $\vec{\alpha}$

i.e. $\nabla_{\vec{\alpha}'(t)} \vec{\alpha}'(t) = 0$

$\left(\text{or } \frac{D\vec{\alpha}'(t)}{dt} = 0\right)$

Remark: By Th$^{\underline{m}}$ 7, $\|\vec{\alpha}'(t)\|$ must be constant (but not necessarily 1)

we call an unparameterized curve a geodesic if its arc length parameterization is a geodesic