Idea: if pendulum is very long then its motion (almost) gives a tangent vector to the sphere (its motion is (almost) a path on sphere)

we think of the earth as not moving but we transport the pendulum around the lattitude circle so we get back to the start in 24 hours

since we are moving slowly and applying no "force" to the pendulum, we expect its swing to "not change" or be "parallel" as we move

thus above computation shows the plane in which the pendulum swings will rotate by $2\pi \cos u_0$ so the "speed" of rotation is

$$2\pi \cos u_0 = 2\pi \cos\left(\frac{\pi}{2} - \phi\right) = 2\pi \sin\phi \ \text{radians}/\text{day}$$

C. Geodesics

we call a regular parameterized curve

$$\vec{\alpha} : [a,b] \to \Sigma$$

a geodesic if its tangent vector is parallel along $\vec{\alpha}$

i.e. $\nabla_{\vec{\alpha}'(t)} \vec{\alpha}'(t) = 0$

(or $\frac{D\vec{\alpha}'(t)}{dt} = 0$)

Remark: By Th$^{\underline{m}}$ 7, $\|\vec{\alpha}'(t)\|$ must be constant (but not necessarily 1)

we call an unparameterized curve a geodesic if its arc length parameterization is a geodesic

Remark: $\frac{D\vec{\alpha}'(t)}{dt}$ is the part of $\vec{\alpha}''(t)$ that is tangent to Σ

so it is the "acelleration" of $\vec{\alpha}(t)$ as seen on Σ

if $\vec{\alpha}$ in $\mathbb{R}^2$ (xy-plane), then

$$\frac{D\vec{\alpha}'(t)}{dt} = \vec{\alpha}''(t)$$

so $\vec{\alpha}$ is a geodesic in the xy-plane

iff

$\vec{\alpha}$ is a straight line!

$$\boxed{\text{Geodesics will generalize straight lines to all Riemannian surfaces}}$$

Th$^{\underline{m}}$ 9:

let Σ be a surface and g be a Riemannian metric on Σ

given a point $\vec{p} \in \Sigma$ and vector $\vec{v} \in T_{\vec{p}}\Sigma$ with $\|v\|_g = 1$

There is a unique geodesic

$$\vec{\gamma} : (-\varepsilon, \varepsilon) \to \Sigma$$

such that $\vec{\gamma}(0) = \vec{p}$ and $\vec{\gamma}'(0) = \vec{v}$

Proof:

In local coordinates $\vec{f} : V \to \Sigma$ around $\vec{p}$

we can assume $\vec{f}(0,0) = \vec{p}$

let (v_1, v_2) be such that $D\vec{f}_{(0,0)}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v} = v_1\vec{f}_u + v_2\vec{f}_v$

a path $\vec{\beta} : (-\varepsilon, \varepsilon) \to V$ can be written

$$\vec{\beta}(t) = (a(t), b(t))$$

set $\vec{\alpha}(t) = \vec{f} \circ \vec{\beta}(t) = \vec{f}(a(t), b(t))$

$\vec{\alpha}$ is a geodesis iff $\nabla_{\vec{\alpha}'(t)}\vec{\alpha}'(t) = 0$

now $\vec{\alpha}'(t) = \vec{f}_u a' + \vec{f}_v b'$

from the proof of lemma 6 we see

$$\nabla_{\vec{\alpha}'(t)}\vec{\alpha}'(t) = (a'' + \Gamma_{11}^1 a'a' + \Gamma_{12}^1 a'b' + \Gamma_{21}^1 b'a' + \Gamma_{22}^1 b'b')\vec{f}_u$$
$$+ (b'' + \Gamma_{11}^2 a'a' + \Gamma_{12}^2 a'b' + \Gamma_{21}^2 b'a' + \Gamma_{22}^2 b'b')\vec{f}_v$$

so $\vec{\alpha}$ is a geodesic iff

$$(*)\quad \boxed{\begin{aligned} a'' + \Gamma_{11}^1(a')^2 + 2\Gamma_{12}^1 a'b' + \Gamma_{22}^1(b')^2 &= 0 \\ b'' + \Gamma_{11}^2(a')^2 + 2\Gamma_{12}^2 a'b' + \Gamma_{22}^2(b')^2 &= 0 \end{aligned}}$$

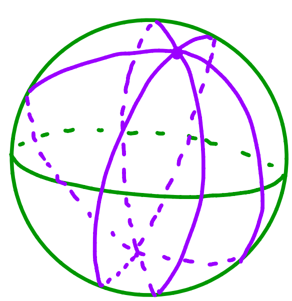
this is a system of second order ordinary differential equations. So standard ODE theory says there is a unique sol$^{\underline{n}}$ given initial conditions

$$a(0) = 0 \qquad a'(0) = v_1$$
$$b(0) = 0 \qquad b'(0) = v_2$$

example:

$S^2 \subset \mathbb{R}^3$ earlier we saw any great circle $C \subset S^2$ is a geodesic

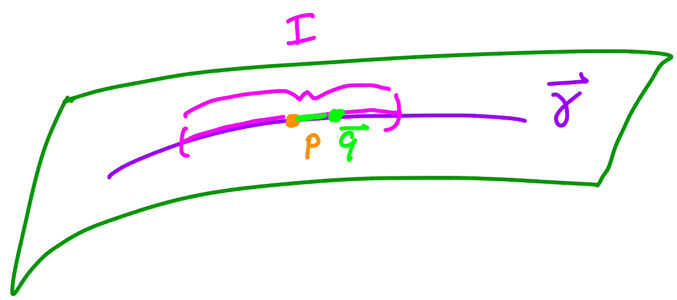
note: uniqueness in Th$^{\underline{m}}$ 9 says any geodesic is (part of) a great circle



since any direction at any point is tangent to a great circle.

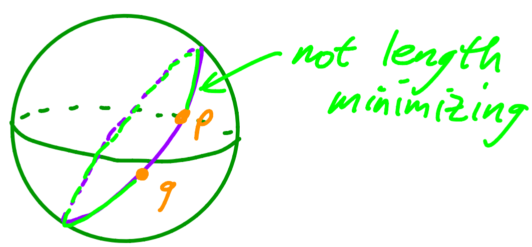
Thm 10:

> let Σ be a surface with a Riemannian metric g
> for any point $\vec{p}$ in Σ and geodesic $\vec{\gamma}$ through $\vec{p}$
> There is an interval I about $\vec{p}$ in $\vec{\gamma}$ such that
> for any point $\vec{q}$ in I, $\vec{\gamma}$ is the shortest path
> from $\vec{p}$ to $\vec{q}$
>
> "geodesics are locally length-minimizing"



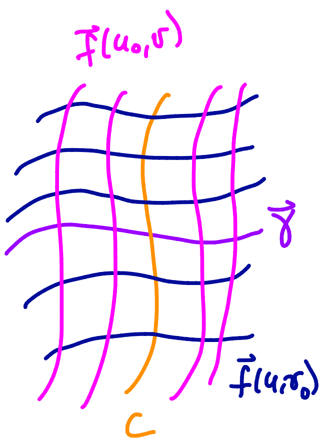
Remark: Only locally length-minimizing

e.g. S^2



Proof: let C be a curve through $\vec{p}$ orthogonal to $\vec{\gamma}$

Claim 1:



> There is a coordinate system
> $\vec{f}: V \to \Sigma$ around $\vec{p}$
> such that
> 1) $\vec{f}(0,0) = \vec{p}$
> 2) $\vec{f}(u,v_0)$ is a geodesic orthogonal to C for any v_0
> (so $\vec{f}(u,0) = \vec{\gamma}(u)$)
> 3) $\vec{f}(u_0,v)$ is a curve orthogonal to each $\vec{f}(u,v_0)$

assuming the claim for now, note in these coordinates

$$\vec{f}_u \cdot \vec{f}_v = 0 \quad (\text{from } 3))$$

$$\text{so } g_{12} = g_{21} = 0$$

$\vec{\gamma}'(u) = \vec{f}_u(u,0)$

so $\|\vec{\gamma}'\|_g = \sqrt{\vec{f}_u \cdot \vec{f}_u} = \sqrt{g_{11}}$

for any path $\vec{\alpha}(t)$ from $\vec{p}$ to $\vec{q}$ we can write it

$$\vec{f}(u(t), v(t)) \qquad a \leq t \leq b$$

for some $u(t), v(t)$

now $\vec{\alpha}'(t) = \vec{f}_u u'(t) + \vec{f}_v v'(t)$

and $\|\vec{\alpha}'(t)\| = \sqrt{\vec{f}_u \cdot \vec{f}_u (u')^2 + \vec{f}_v \cdot \vec{f}_v (v')^2}$

recall $g_{12} = 0 = g_{21}$

$$= \sqrt{g_{11}(u')^2 + g_{22}(v')^2} \geq \sqrt{g_{11}}\,|u'(t)|$$

since $g_{22} \geq 0$

Claim 2:

$$\boxed{\begin{array}{l} g_{11}(u,v) \text{ is independent of } v \\ \text{i.e. } g_{11}(u) \end{array}}$$

so $\text{length}(\vec{\alpha}) = \int_a^b \|\vec{\alpha}'(t)\|\,dt$

$$\geq \int_a^b \sqrt{g_{11}(u(t))}\,|u'(t)|\,dt$$

$$\left(\begin{array}{ll} u = u(t) & u(a) = 0 \\ du = u'(t)\,dt & u(b) = u_0 \\ & \text{where } \vec{f}(0,u_0) = \vec{q} \\ \text{(upto sign)} & \end{array} \right)$$

$$= \int_0^{u_0} \sqrt{g(u)}\,du$$

$$= \text{length}(\vec{\gamma} \text{ from } \vec{p} \text{ to } \vec{q})$$

so done given claims!

Proof of Claim 2: recall the geodesic equations are

$$u'' + \Gamma_{11}^{1}(u')^2 + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}(v')^2 = 0$$

$$v'' + \Gamma_{11}^{2}(u')^2 + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}(v')^2 = 0$$

we know $\vec{f}(u, v_0)$ are geodesics
so the second equation is

$$\underset{\substack{\parallel \\ \Gamma_{11}^{2}}}{\Gamma_{11}^{2}}(u')^2 = 0$$

from lemma 2

$$\Gamma_{11}^{2} = g^{21}\tfrac{1}{2}\big((g_{11})_u + (g_{11})_u - (g_{11})_u\big)$$

$$+\, g^{11}\tfrac{1}{2}\big((g_{12})_u + (g_{12})_u - (g_{11})_v\big)$$

we know $\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11} & \cancelto{0}{g_{12}} \\ \cancelto{0}{g_{12}} & g_{22} \end{pmatrix}^{-1}$

so $g^{21} = 0$

and $0 = \Gamma_{11}^{2} = -\tfrac{1}{2}g^{11}(g_{11})_v$

so $(g_{11})_v = 0$ and g_{11} is independent of v ✓

Claim 1 uses some ODE theory. We might prove it later
but for now see Shifrin's notes (Thm 3.3 in Appendix)

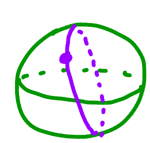
Remark: So geodesics are like "lines" in $\mathbb{R}^n$ in that

① "2nd derivative" is zero

$$\left(\frac{D\vec{\gamma}'(t)}{dt} = \nabla_{\vec{\alpha}'(t)}\vec{\alpha}'(t) = 0\right)$$

② "locally" distance minimizing

but unlike "lines" in that they

① can return to starting point 

② can have many geodesics through two points (in $\mathbb{R}^n$ only one)

example:

$\Sigma = \{(x,y,z) \mid x^2+y^2=1\}$

$\vec{F}(u,v) = (\cos u, \sin u, v)$

so $$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\vec{F}(u(t), v(t))$ gives a geodesic

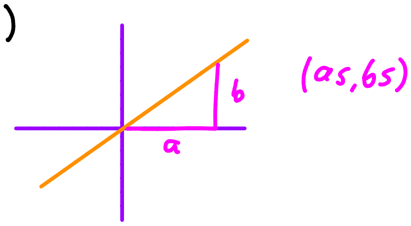
iff

$u''(t) = 0$

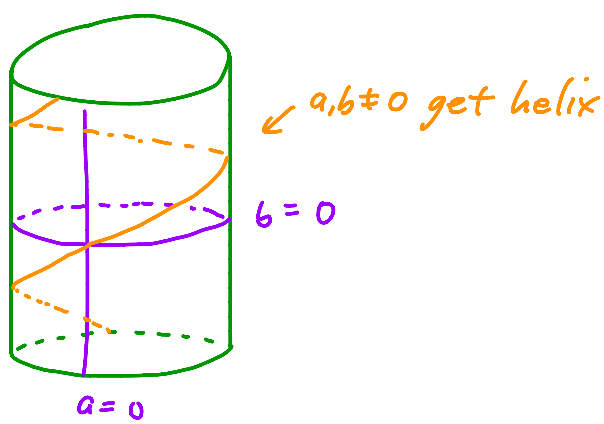
$v''(t) = 0$

i.e. if $(u(t), v(t))$ gives a line in $\mathbb{R}^2$ (param by arc length) then $\vec{F}(u(t), v(t))$ is a geodesic in Σ

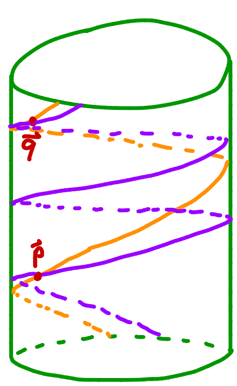
e.g. $\vec{\alpha}(s) = (\cos as, \sin as, bs)$



so we get



note:



if $\vec{p}, \vec{q}$ not on same "circle" then there are infinitely many geodesics connecting them let's consider one more thing about curves on a surface and geodesics

let $\Sigma \subset \mathbb{R}^3$ be a surface and

$\vec{\alpha}$ an arc length parameterization of a curve $C \subset \Sigma$

let $\vec{N}$ denote a unit normal vector to Σ

<u>note:</u> 1) $\vec{\alpha}'(s)$ is tangent to Σ and

2) $\vec{N}$ and $\vec{N} \times \vec{\alpha}'(s)$ are linearly independent and span the orthogonal complement to $\vec{\alpha}'(s)$

(they are also orthonormal)

3) $\vec{\alpha}''(s)$ is perpendicular to $\vec{\alpha}'(s)$ so

$$\vec{\alpha}''(s) = \underbrace{\left[\vec{\alpha}''(s) \cdot (\vec{N} \times \vec{\alpha}'(s))\right]}_{\text{tangential component of } \vec{\alpha}''(s)\text{, we will call this the geodesic curvature}} \vec{N} \times \vec{\alpha}'(s) + \underbrace{\left[\vec{\alpha}''(s) \cdot \vec{N}\right]}_{\text{normal component of } \vec{\alpha}''\text{, normal curvature } K_n \text{ studied earlier}} \vec{N}$$

we denote the geodesic curvature by

$$K_g(s) = \vec{\alpha}''(s) \cdot (\vec{N} \times \vec{\alpha}'(s))$$

$$\left(= \| \nabla_{\vec{\alpha}'(s)} \vec{\alpha}'(s) \| \right)$$

this is the part of the "accelleration" (ie $\vec{\alpha}''(s)$) you can feel/see in Σ

4) since $\vec{N} \times \vec{\alpha}'$ and $\vec{N}$ are orthonormal

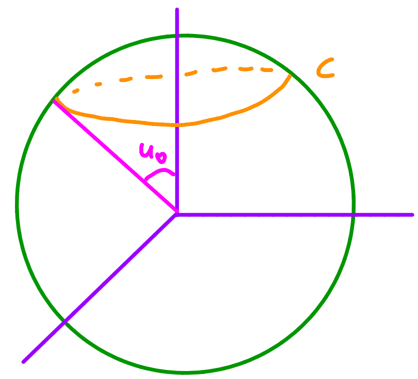
$$\|\vec{\alpha}''\|^2 = K_g^2 + K_n^2$$

Clearly

$$\boxed{K_g = 0 \Leftrightarrow C \text{ is a geodesic}}$$

example:

let C be the curve on S^2 of "lattitude" u_0 studied above



so C is parameterized by

$$(\sin u_0 \cos v, \sin u_0 \sin v, \cos u_0)$$

with $0 \le v \le 2\pi$

a unit speed parameterization is

$$\vec{\alpha}(s) = \left(\sin u_0 \cos\left(\frac{s}{\sin u_0}\right), \sin u_0 \sin\left(\frac{s}{\sin u_0}\right), \cos u_0\right)$$

so

$$\vec{\alpha}'(s) = \left(-\sin\left(\frac{s}{\sin u_0}\right), \cos\frac{s}{\sin u_0}, 0\right)$$

and

$$\vec{\alpha}''(s) = \left(-\frac{1}{\sin u_0}\cos\left(\frac{s}{\sin u_0}\right), -\frac{1}{\sin u_0}\sin\frac{s}{\sin u_0}, 0\right)$$

so $\|\vec{\alpha}''(s)\| = \dfrac{1}{\sin u_0}$

also $K_n = \vec{\alpha}'' \cdot \underbrace{\vec{\alpha}}_{=\vec{N}} = -\left(\cos^2\left(\frac{s}{\sin u_0}\right) + \sin^2\left(\frac{s}{\sin u_0}\right)\right) = -1$

so $\dfrac{1}{\sin^2 u_0} = K_g^2 + K_n^2 = K_g^2 + 1$

and $K_g^2 = \dfrac{1-\sin^2 u_0}{\sin^2 u_0} = \dfrac{\cos^2 u_0}{\sin^2 u_0} = \cot^2 u_0$

We saw from Claim 1 in proof of Th$^{\underline{m}}$ 10 that we can find a coordinate chart

$$\vec{f} : V \to \Sigma$$

for a surface such that

$$g = \begin{pmatrix} g_{11}(u,v) & 0 \\ 0 & g_{22}(u,v) \end{pmatrix}$$

at each point $\vec{f}(u,v)$ we have the orthonormal basis for $T_{\vec{f}(u,v)}\Sigma$

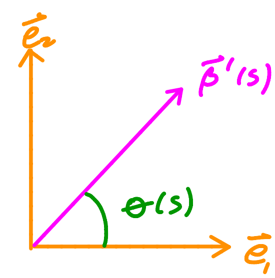
$$\vec{e}_1(u,v) = \frac{\vec{f}_u(u,v)}{\sqrt{g_{11}(u,v)}} \quad , \quad \vec{e}_2(u,v) = \frac{\vec{f}_v(u,v)}{\sqrt{g_{22}(u,v)}}$$

now given a curve parameterized by arc length

$$\vec{\beta}(s) = \vec{f}(u(s), v(s))$$

let $\theta(s)$ = angle between $\vec{e}_1$ and $\vec{\beta}'(s)$

$$= \cos^{-1}\left(\vec{e}_1(u(s), v(s)) \cdot \vec{\beta}'(s)\right)$$



Remark: Recall when studying curves in $\mathbb{R}^2$ we also had an angle $\theta(s)$ between $\vec{\alpha}'$ and x-axis from this we could compute curvature and learned a lot about curves the above $\theta(s)$ is the analog of this in surfaces

Th$^{\underline{m}}$ 11:

With the notation above

$$\kappa_g(s) = \frac{d\theta}{ds} + \left(\nabla_{\vec{\beta}'(s)} \vec{e}_1\right) \cdot \vec{e}_2$$

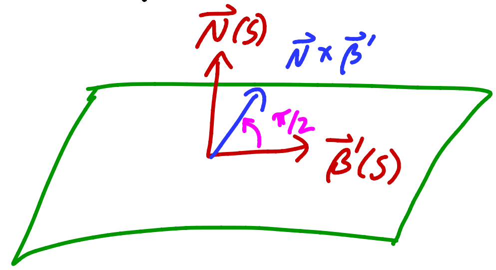
$$= \frac{d\theta}{ds} + \frac{1}{2\sqrt{g_{11} g_{22}}} \left\{ -(g_{11})_v u' + (g_{22})_u v' \right\}$$

Proof: recall $\kappa_g = \vec{\beta}''(s) \cdot (\vec{N} \times \vec{\beta}'(s)) = (\nabla_{\vec{\beta}'(s)} \vec{\beta}'(s)) \cdot (\vec{N} \times \vec{\beta}'(s))$

since this is the tangential component of $\vec{\beta}''(s)$

we know $\vec{\beta}'(s) = (\cos\theta(s))\vec{e}_1 + (\sin\theta(s))\vec{e}_2$

and $\vec{N} \times \vec{\beta}'(s)$ is $\vec{\beta}'(s)$ rotated $\frac{\pi}{2}$ counterclockwise



so $\vec{N} \times \vec{\beta}'(s) = (-\sin\theta(s))\vec{e}_1 + (\cos\theta(s))\vec{e}_2$

thus

$$\kappa_g = (\nabla_{\vec{\beta}'(s)} \vec{\beta}'(s)) \cdot (-(\sin\theta(s))\vec{e}_1 + (\cos\theta(s))\vec{e}_2)$$

$$= \left[\nabla_{\vec{\beta}'(s)}\left((\cos\theta(s))\vec{e}_1 + (\sin\theta(s))\vec{e}_2\right)\right] \cdot (-(\sin\theta(s))\vec{e}_1 + (\cos\theta(s))\vec{e}_2)$$

$$\overset{\circledast}{=} \left[\cos(\theta(s))\nabla_{\vec{\beta}'(s)}\vec{e}_1 + (\sin\theta(s))\nabla_{\vec{\beta}'(s)}\vec{e}_2\right] \cdot (-(\sin\theta(s))\vec{e}_1 + (\cos\theta(s))\vec{e}_2)$$

$$+ \left[-\theta'(s)(\sin\theta(s))\vec{e}_1 + \theta'(s)(\cos\theta(s))\vec{e}_2\right] \cdot (-(\sin\theta(s))\vec{e}_1 + (\cos\theta(s))\vec{e}_2)$$

$$= -\cos\theta\sin\theta(\nabla_{\vec{\beta}'}\vec{e}_1)\cdot\vec{e}_1 + \cos^2\theta(\nabla_{\vec{\beta}'(s)}\vec{e}_1)\cdot\vec{e}_2 - \sin^2\theta(\nabla_{\vec{\beta}'(s)}\vec{e}_2)\cdot\vec{e}_1 + \cos\theta\sin\theta(\nabla_{\vec{\beta}'(s)}\vec{e}_2)\cdot\vec{e}_2$$

$$+ \underbrace{(\theta'\sin^2\theta + \theta'\cos^2\theta)}_{\theta'(s)}$$

for $\circledast$ we need a product rule

exercise:

If h is a function and $\vec{v}$ is a vector field along a path $\vec{\alpha}$

then
$$\nabla_{\vec{\alpha}'}(h\vec{v}) = h'\vec{v} + h\nabla_{\vec{\alpha}'}\vec{v}$$

Hint: tangential component of $(h\vec{v})'$

from above we have

$$\kappa_g = \theta' - \cos\theta\sin\theta\,\underline{(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_1} + \cos^2\theta\,\underline{(\nabla_{\vec\beta'(s)}\vec e_1)\cdot\vec e_2} - \sin^2\theta\,\underline{(\nabla_{\vec\beta'(s)}\vec e_2)\vec e_1}$$
$$+\cos\theta\sin\theta\,\underline{(\nabla_{\vec\beta'(s)}\vec e_2)\cdot\vec e_2}$$

note: $\vec e_1\cdot\vec e_1 = 1$ so as we saw earlier

$$(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_1 + \vec e_1\cdot(\nabla_{\beta'}\vec e_1) = 0$$
$$\parallel$$
$$2(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_1$$

so $\underline{(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_1 = 0}$

similarly $\underline{(\nabla_{\vec\beta'}\vec e_2)\cdot\vec e_2 = 0}$

also $\vec e_1\cdot\vec e_2 = 0$ so

$$(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2 + \vec e_1\cdot(\nabla_{\beta'}\vec e_2) = 0$$
$$\therefore\ \underline{(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2} = \underline{-\ \vec e_1\cdot(\nabla_{\beta'}\vec e_2)}$$

and we have

$$\kappa_g = \theta' + (\cos^2\theta + \sin^2\theta)(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2$$
$$= \theta' + (\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2$$

thus verifying first equation

to finish we need to see

$$(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2 = \frac{1}{2\sqrt{g_{11}g_{22}}}\left(-(g_{11})_v u' + (g_{22})_u v'\right)$$

now $(\nabla_{\vec\beta'}\vec e_1)\cdot\vec e_2 = \left(\frac{d}{dt}\vec e_1\right)^T\cdot\vec e_2$ tangential component

$$= \Big(\underbrace{\tfrac{d}{dt}\vec e_1 - \left[\left(\tfrac{d}{dt}\vec e_1\right)\cdot\vec N\right]\vec N}\Big)\cdot\vec e_2$$

$$= \left(\frac{d}{dt}\vec e_1\right)\cdot\vec e_2$$

$$= \frac{d}{dt}\left(\frac{\vec{f}_u}{\sqrt{g_{11}}}\right) \cdot \frac{\vec{f}_v}{\sqrt{g_{22}}}$$

$$= \left[\frac{1}{\sqrt{g_{11}}}\left(\vec{f}_{uu}\, u' + \vec{f}_{uv}\, v'\right) + \left(\frac{1}{\sqrt{g_{11}}}\right)' \vec{f}_u\right] \cdot \frac{\vec{f}_v}{\sqrt{g_{22}}}$$

$$= \frac{1}{\sqrt{g_{11} g_{22}}}\left(\left(\Gamma_{11}^{1}\vec{f}_u + \Gamma_{11}^{2}\vec{f}_v\right)u' + \left(\Gamma_{12}^{1}\vec{f}_u + \Gamma_{12}^{2}\vec{f}_v\right)v'\right)\cdot \vec{f}_v$$

$$= \frac{1}{\sqrt{g_{11} g_{22}}}\left(g_{22}\left(\Gamma_{11}^{2} u' + \Gamma_{12}^{2} v'\right)\right)$$

recall $\Gamma_{11}^{2} = \cancelto{0}{g^{21}}\frac{1}{2}\left(\text{——}\right) + g^{22}\frac{1}{2}\left((g_{12})_u + (g_{12})_u - (g_{11})_v\right)$

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{11} & \cancelto{0}{g_{12}} \\ \cancelto{0}{g_{21}} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g_{11} g_{22}}\begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix}$$

so $g^{22} = \dfrac{g_{11}}{g_{11} g_{22}} = \dfrac{1}{g_{22}}$

thus $\Gamma_{11}^{2} = -\dfrac{1}{2g_{22}}(g_{11})_v$

similarly you can check

$$\Gamma_{12}^{2} = \frac{1}{2g_{22}}(g_{22})_u$$

so $(\nabla_{\vec{\beta}'}\vec{e}_1)\cdot\vec{e}_2 = \dfrac{1}{\sqrt{g_{11} g_{22}}}\left(g_{22}\left(-\dfrac{1}{2g_{22}}(g_{11})_v\, u' + \dfrac{1}{2g_{22}}(g_{22})_u\, v'\right)\right)$

$$= \frac{1}{2\sqrt{g_{11} g_{22}}}\left(-(g_{11})_v\, u' + (g_{22})_v\, u'\right)$$