# VI Curvature and Surfaces

## A. Classification of Surfaces

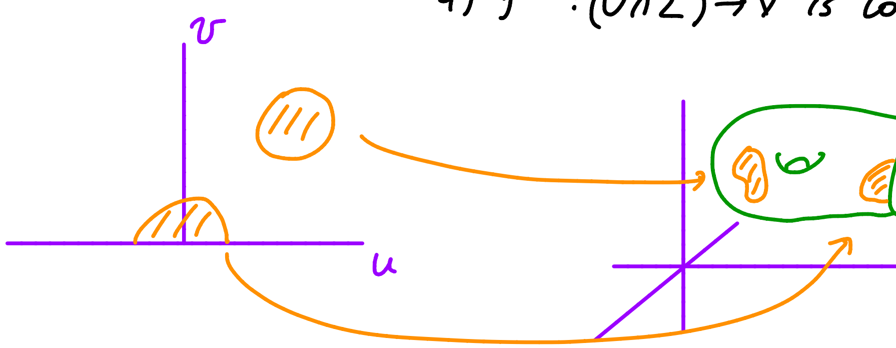
We first introduce surfaces with boundary

a subset $\Sigma \subset \mathbb{R}^N$ (some $N$) is a surface with boundary if for each point $\vec{p} \in \Sigma$ we have

- an open set $V$ in $\mathbb{R}^2_+ = \{(u,v) \mid v \geq 0\}$
- an open set $U$ in $\mathbb{R}^N$ containing $\vec{p}$
- a differentiable map

$$\vec{f} : V \to U$$

such that
1) $\vec{f}$ is injective
2) $\text{image}(\vec{f}) = \Sigma \cap U$
3) $\text{Rank}(D\vec{f}_{\vec{p}}) = 2 \quad \forall \vec{q} \in V$
4) $\vec{f}^{-1} : (U \cap \Sigma) \to V$ is continuous




we say $\partial\Sigma = \{\vec{p} \in \Sigma$ such that $\vec{p}$ is in the image of $u$-axis in some coord chart$\}$

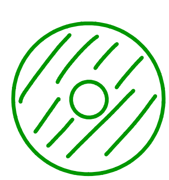
$\uparrow$ boundary

example:

1) any surface $\Sigma$ is a surface with boundary, just $\partial\Sigma = \emptyset$
2) $D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$

$\partial D^2 = S^1$

3) $A = \{(x,y) \mid a \leq x^2+y^2 \leq 1\}$



$\partial A = S^1 \cup S^1$

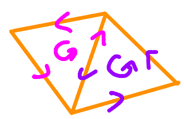
given a surface $\Sigma$ (maybe with boundary) a triangulation of $\Sigma$ is a collection $\mathcal{T} = \{\Delta_i\}_{i=1}^{n}$ of subsets of $\Sigma$ such that

1) $\Sigma = \bigcup_{i=1}^{n} \Delta_i$

2) each $\Delta_i$ is the image of a triangle under an (orientation preserving) local coordinate chart

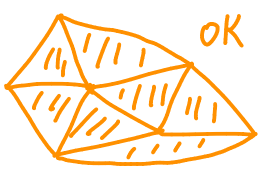
3) $\Delta_i \cap \Delta_j = \begin{cases} \emptyset \\ \text{single vertex} \\ \text{single edge} \end{cases}$ $\quad i \neq j$

if edge, then orientation on edge from $\Delta_i$ opposite the one from $\Delta_j$



4) at most one edge of $\Delta_i$ can be in the boundary of $\Sigma$

example:

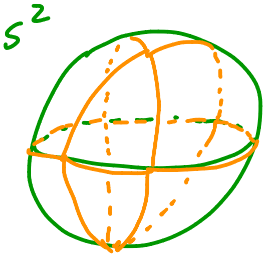


OK

not ok

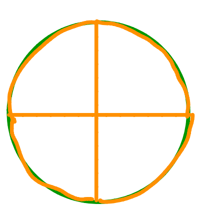
not ok

$S^2$

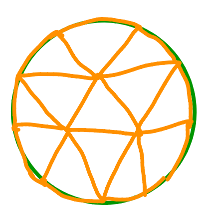


8 triangles

$D^2$



or



Fact (Radó, 1925):

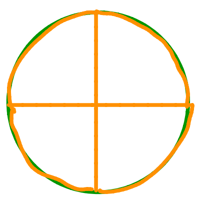
$\boxed{\text{all compact surfaces have triagulation.}}$

given a surface $\Sigma$ and a triangulation $\mathcal{T}$ of $\Sigma$ the <u>Euler characteristic</u> is

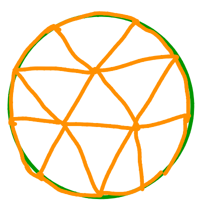
$$\chi(\Sigma) = (\text{number of verticies of } \mathcal{T}) - (\text{number of edges of } \mathcal{T}) + (\text{number of faces of } \mathcal{T})$$

<u>examples:</u>

$D^2$



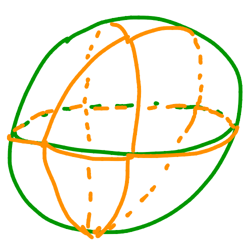
$\chi = 5 - 8 + 4 = 1$



$\chi = 12 - 24 + 13 = 1$

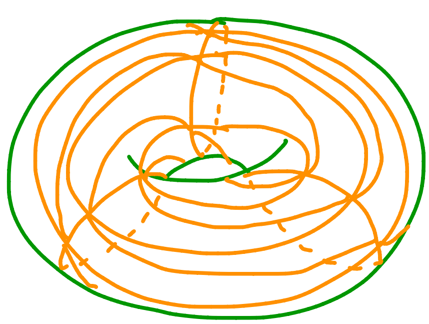
<u>note:</u> $\chi$ same for both triaingulations

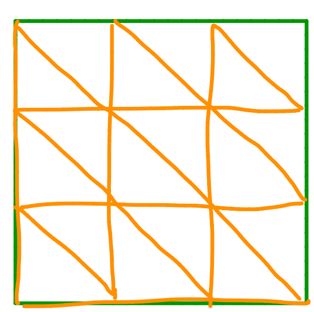
$S^2$



$\chi = 6 - 12 + 8 = 2$

$T^2$





$\chi = 9 - 27 + 18 = 0$

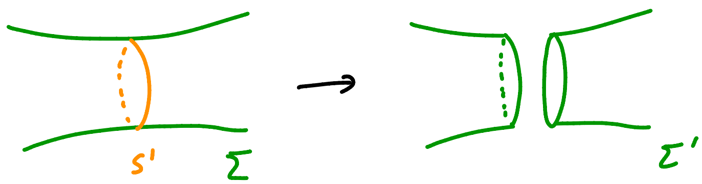
## $Th^{\underline{m}}$ 1:

$\boxed{\text{the Euler characteristic of a surface is independent of the triangulation}}$

we prove this later using geometry!

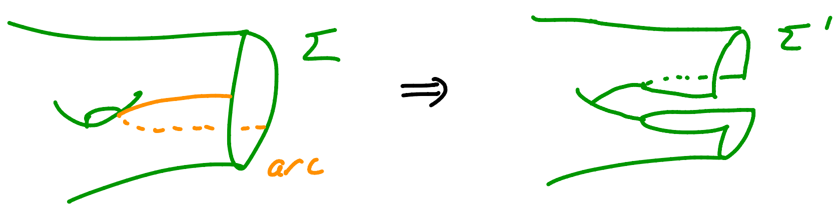
## Facts about Euler characteristic:

if you get $\Sigma'$ from $\Sigma$ by cutting on a circle




then $\chi(\Sigma') = \chi(\Sigma)$

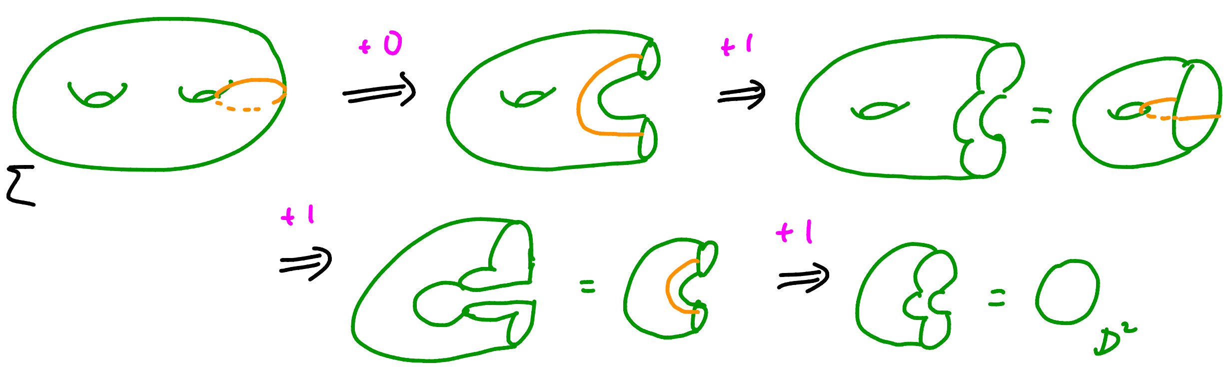
if you get $\Sigma'$ from $\Sigma$ by cutting on an arc (with $\partial$ in $\partial\Sigma$)




then $\chi(\Sigma') = \chi(\Sigma) + 1$

from this, and $\chi(D^2)=1$, you can compute $\chi$ of any surface

e.g.




$\chi = 1$

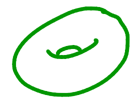
so $\chi(\Sigma) = 1 - 3 = -2$

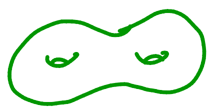
## Topology Facts:

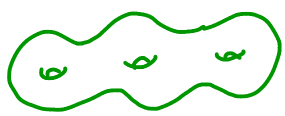
"is" means "homeomorphic to", ie $X, Y$ are homeomorphic if $\exists$ a continuous bijection $f: X \to Y$ with continuous inverse

(I) A compact connected oriented surface is one of the following

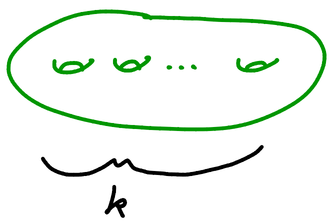
$\Sigma_0 = S^2$ 2-sphere 

$\Sigma_1 = T^2 = S^1 \times S^1$ torus 

$\Sigma_2 =$  2 holed torus

$\Sigma_3 =$  3 holed torus

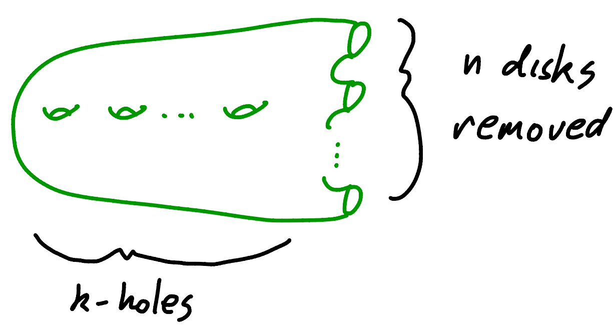
$\vdots$

$\Sigma_k =$  k holed torus

$\vdots$

(II) $\chi(\Sigma_k) = 2 - 2k$

(III) a compact connected oriented surface with boundary is obtained from one of the surfaces above by removing some number of disks

$\Sigma_{k,n} =$ 


(IV) $\chi(\Sigma_{k,n}) = 2 - 2k - n$

note: Facts imply two surfaces are the same (homeomorphic)

$\Longleftrightarrow$

they have same $\chi$ and number of boundary components

## B. Gauss-Bonnet Theorem

$\Sigma$ a surface with Riemannian metric $g$

$\vec{\alpha} : [0,l] \to \Sigma$ is a simple closed piecewise smooth curve

if 1) $\vec{\alpha}(0) = \vec{\alpha}(l)$

2) $\vec{\alpha}$ is one-to-one on $[0,l)$

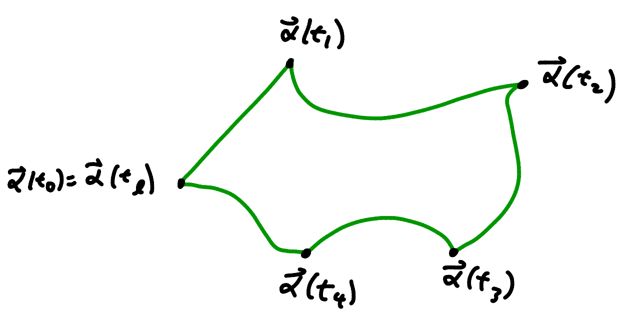
(i.e. $t_1 \neq t_2 \in [0,l) \Rightarrow \vec{\alpha}(t_1) \neq \vec{\alpha}(t_2)$)

3) there are points

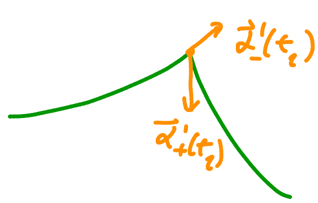
$$0 = t_0 < t_1 < \dots < t_k = l$$

such that

$\vec{\alpha}$ is regular on $[t_{i-1}, t_i]$ for $i = 1, \dots, k$




note: at $t_i$ there are 2 "tangent" vectors

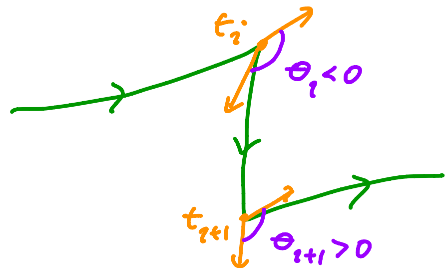



$$\vec{\alpha}'_\pm(t_i) = \lim_{\substack{h \to 0 \\ \pm h > 0}} \frac{\vec{\alpha}(t_i + h) - \vec{\alpha}(t_i)}{h}$$

the angle $\theta_i$ = angle from $\vec{\alpha}'_-(t_i)$ to $\vec{\alpha}'_+(t_i)$

is called the exterior angle at $t_i$

eg




as at the end of the last section let

$$\vec{f} : V \to \Sigma$$

be a coordinate chart such that

$$g = \begin{pmatrix} g_{11}(u) & 0 \\ 0 & g_{22}(u,v) \end{pmatrix}$$

and set $\vec{e}_1(u,v) = \dfrac{\vec{f}_u(u,v)}{\sqrt{g_{11}(u)}}$ , $\vec{e}_2 = \dfrac{\vec{f}_v(u,v)}{\sqrt{g_{22}(u,v)}}$

for a curve $\vec{\alpha}$ and point $t'$ where $\vec{\alpha}$ is regular we let

$$\theta(t') = \text{angle between } \vec{e}_1 \text{ and } \vec{\alpha}(t')$$
$$= \cos^{-1}\left(\frac{\vec{e}_1 \cdot \vec{\alpha}'(t')}{\|\vec{\alpha}'(t')\|}\right)$$

**Th$^{m}$2 (Turning tangents):**

with notation as above, if $\vec{\alpha}$ bounds a disk in a coordinate chart

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \theta'(t)\,dt + \sum_{i=1}^{k} \theta_i = \pm 2\pi$$

if $\theta_{\pm}(t_i) = \text{angle between } \vec{\alpha}'_{\pm}(t_i) \text{ and } \vec{e}_1$

then this can be written

$$\sum_{i=1}^{k} \left(\theta_-(t_i) - \theta_+(t_{i-1})\right) + \sum_{i=1}^{k} \theta_i = \pm 2\pi$$

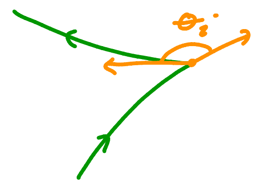
# Sketch of Proof:

Recall in Corollary II.11 we saw a simple (embedded) regular curve $\vec{\beta}$ in $\mathbb{R}^2$ satisfied

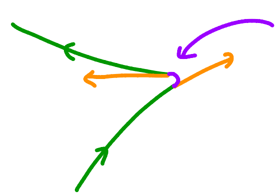
$$\text{Total signed curvature} = \int_0^{\ell} \kappa_\sigma(s)\,ds$$
$$= \int_0^{\ell} \theta'(s)\,ds = \pm 2\pi$$

where $\theta$ = angle between $\vec{\beta}'(s)$ and positive $x$-axis

now if $\vec{\beta}$ has corners




note you could "round" the corner to get a new $\vec{\beta}$ that is the same as the old one except near $\vec{\beta}(t_i)$ where it has been "smoothed"




and the new part adds $\theta_i$ to integral $\int \theta'(s)\,ds$

it is not too hard to turn this into a proof for curves in $\mathbb{R}^2$ with $\vec{e}_1 = \frac{\partial}{\partial x}$ and $\vec{e}_2 = \frac{\partial}{\partial y}$

Thm 2 just says this also works in general coordinate charts

(this is a hard proof, but done by Hopf in 1938)

## Th$^{m}$ 3 (Local Gauss-Bonnet Th$^{m}$):

let $\Sigma$ be a surface with Riemannian metric $g$

$\vec{\alpha}: [0,\ell] \to \Sigma$ an arc-length parameterized simple piecewise regular curve whose image in $\Sigma$ bounds a region $R \subset \Sigma$ such that

1) $R$ is contained in a orthonormal coordinate chart (i.e. one like above)
2) $R$ is a disk
   (i.e. there is a continuous bijection to $D^2$ with a continuous inverse)
3) $\vec{\alpha}$ is oriented counter clockwise in the coordinate chart


with the notation above

$$\Big(\sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \underbrace{\kappa_g(s)}_{\text{geodesic curvature}}\, ds\Big) + \int_R \underbrace{K}_{\text{Gauss curvature}}\, dA + \sum_{i=1}^{k} \theta_i = 2\pi$$

**Proof:** from Th$^{m}$ I.11 we have

$$\kappa_g(s) = \frac{d\theta}{ds} + \frac{1}{2\sqrt{g_{11}g_{22}}}\left\{-(g_{11})_v u' + (g_{22})_u v'\right\}$$

where $\vec{\alpha}(s) = \vec{f}(u(s), v(s))$

integrating gives

$$\sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \kappa_g(s)\, ds = \overbrace{\sum_{i=1}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{g_{11}g_{22}}}\left\{-(g_{11})_v u' + (g_{22})_u v'\right\} ds}^{\circledast}$$

$$+ \underbrace{\sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} \frac{d\theta}{ds}\, ds}_{= 2\pi - \sum_{i=1}^{k} \theta_i \text{ by Th}^{m}\, 2}$$

left to study ⊛

Recall from $Th^{\underline{m}}$ V.1 in orthogonal coordinates

$$K = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left\{\left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right)_v + \left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)_u\right\}$$

Recall from Calculus III that for functions $P, Q : \mathbb{R} \to \mathbb{R}$

Green's $Th^{\underline{m}}$ says

$$\int_{\partial R} P(u(s), v(s))\, u'(s) + Q(u(s), v(s))\, v'(s)\, ds$$

$$= \int_R \frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\, du\, dv$$

so

$$\int_{\partial R} \frac{1}{2\sqrt{g_{11}g_{22}}}\left(-(g_{11})_v u' + (g_{22})_u v'\right) ds$$

$$= \frac{1}{2}\int_R \left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)_u + \left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right)_v\, du\, dv$$

$$= \int_R \frac{1}{2\sqrt{g_{11}g_{22}}}\left(\left(\frac{(g_{22})_u}{\sqrt{g_{11}g_{22}}}\right)_u + \left(\frac{(g_{11})_v}{\sqrt{g_{11}g_{22}}}\right)_v\right) \underbrace{\sqrt{g_{11}g_{22}}\, du\, dv}$$

recall $dA = \sqrt{\det g}\, du\, dv$

$$= -\int K\, dA$$

and

$$\sum_{i=1}^{k} \int_{s_{i-1}}^{s_i} k_g(s)\, ds + \int_R K\, dA + \sum_{i=1}^{k} \theta_i = 2\pi$$

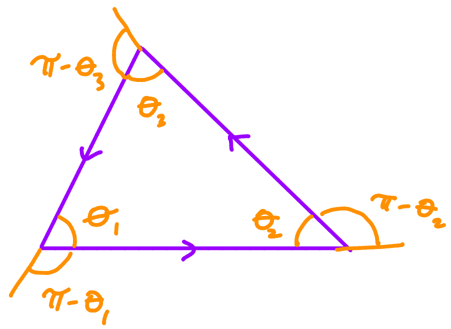
examples:

1) Consider $\Sigma = \mathbb{R}^2$ with $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

recall lines are geodesics and $\kappa_g = 0$ for them

also $K = 0$

so given a triangle




Th$^{\underline{m}}$ 3 says

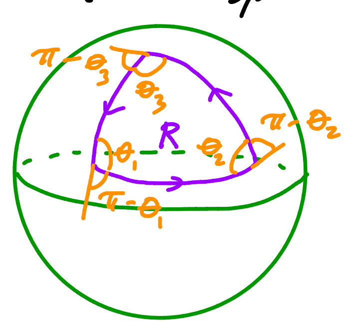
$$0 + 0 + (\pi - \theta_1) + (\pi - \theta_2) + (\pi - \theta_3) = 2\pi$$

so $\boxed{\pi = \theta_1 + \theta_2 + \theta_3}$ nice proof of fact you know!

2) Consider geodesic triangles on $S^2$

edges are geodesics so $\kappa_g = 0$

also $K = 1$ (unit sphere)




Th$^{\underline{m}}$ 3 gives

$$0 + \int_R 1\, dA + [3\pi - (\theta_1 + \theta_2 + \theta_3)] = 2\pi$$

so $(\text{area } R) = (\theta_1 + \theta_2 + \theta_3) - \pi$

note: 1) angle sum in spherical triangle is always $> \pi$

2) the "extra" angle is equal to the area of the triangle!

3) more generally a geodesic triangle on a surface of constant Gauss curvature $K$ has

$$(\theta_1 + \theta_2 + \theta_3) - \pi = K \,\text{Area}(\text{triangle})$$

Th$^{\underline{m}}$ 4 (Gauss-Bonnet):

let $\Sigma$ be a compact, oriented surface with piecewise smooth boundary such that

$$\partial\Sigma = C_0 \cup \ldots \cup C_k$$

where $C_i$ are regular curves

let $g$ be a Riemannian metric on $\Sigma$ and $\theta_i$ the exterior angle between $c_{i-1}$ and $c_i$

Given a triangulation $\mathcal{T}$ of $\Sigma$ with each triagle in an orthonormal coord chart

then we have

$$\boxed{\left(\sum_{i=1}^{k} \int_{C_i} \kappa_g(s)\, ds\right) + \int_\Sigma K \, dA + \sum_{i=1}^{k} \theta_i = 2\pi \chi(\Sigma)}$$

Proof of Thm 1:

left hand side of equation is independent of the triangulation

Cor 5:

If $\Sigma$ is a compact oriented surface with Riemannian metric $g$

then $\int_\Sigma K \, dA = 2\pi \chi(\Sigma)$

Proof: clear

Remark: this is amazing!

e.g. $T^2 \subset \mathbb{R}^4$ with $K = 0$ everywhere

and

$T^2 \subset \mathbb{R}^4$ with very complicated $K$ from earlier

both have $\int_{T^2} K \, dA = 0$

(and so will any other metric)

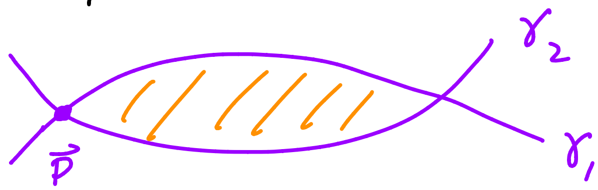
Cor 6:

If $\Sigma$ is a compact oriented surface with a metric having positive Gauss curvature then $\Sigma$ is homeomorphic to $S^2$

Proof: $S^2$ only surface with $\chi > 0$

Cor 7:

If $\Sigma$ is a compact oriented surface with a metric having non-positive Gauss curvature then 2 geodesics $\vec{\gamma}_1, \vec{\gamma}_2$ starting at $\vec{p}$ cannot intersect again in such a way that they bound a disk




Proof: if they did then

$$\underbrace{\sum_{i=1}^{2} \int_{C_i} \kappa_g(s)\,ds}_{=0} + \underbrace{\int_{D^2} K\,dA}_{\leq 0} + \sum_{i=1}^{2} \theta_i = 2\pi \underbrace{\cancel{\chi(D^2)}}_{1}$$

so $\int_{D^2} K\,dA + \theta_1 + \theta_2 = 2\pi$

note: if $\theta_i = \pi$ then $\vec{\gamma}_1$ and $\vec{\gamma}_2$ are tangent at intersection point

but uniqueness of geodesics (Thm V.9)

$\Rightarrow \vec{\gamma}_1 = \vec{\gamma}_2$ in this case

so $\theta_i < \pi$

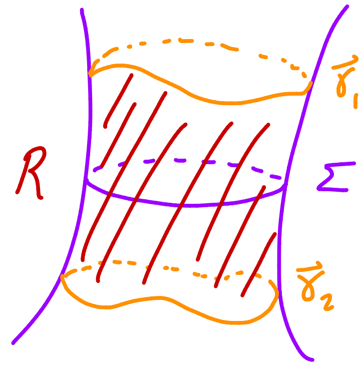
$\therefore\ 2\pi - \theta_1 - \theta_2 > 0$

this contradicts $\int_{D^2} K\,dA < 0$

Cor 8: If $\Sigma$ is homeomorphic to $S^1 \times \mathbb{R}$ (cylinder) with Gauss curvature $K < 0$ then $\Sigma$ has at most one simple closed regular geodesic

**Proof:** Assume $\gamma_1, \gamma_2$ are 2 closed geodesics

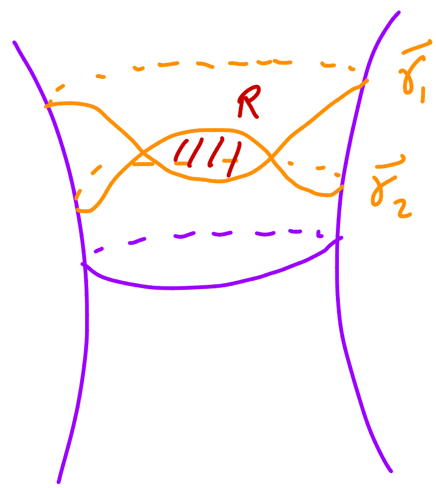
if $\gamma \cap \gamma' = \emptyset$, then they bound a region $R$




$$\sum_{i=1}^{2} \underbrace{\int_{\gamma_i} \kappa_g(s)\,ds}_{\substack{0 \\ \text{(geodesics)}}} + \int_R K\,dA + \underbrace{\sum \theta_i}_{\substack{= \\ 0 \\ \text{(no corners)}}} = 2\pi \underbrace{\chi(R)}_{\substack{= \\ 0}}$$

so $\int_R K\,dA = 0$ $\otimes$ $K < 0$

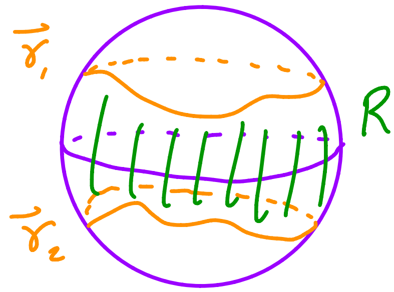
if $\gamma \cap \gamma' \neq \emptyset$, then part of them will bound a region as in Cor 7 $\otimes$ Cor 7




**Cor 9:**

> Any 2 closed simple geodesics $\vec{\gamma}_1, \vec{\gamma}_2$ on a compact oriented surface $\Sigma$ with a metric having <u>positive</u> Gauss curvature <u>must</u> intersect

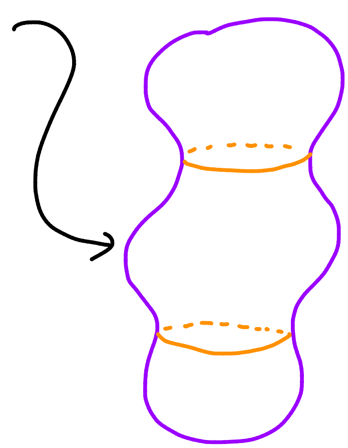
**Proof:** if not, they bound a region $R$ with two boundary components, so $\chi(R) \leq 0$ (recall $\chi(\Sigma_{k,n}) = 2-2k-n$)




so $$\underbrace{\sum \int \kappa_g \, ds}_{=0} + \int_R K \, dA + \underbrace{\sum \theta_i}_{=0} \leq 0$$

this contradicts $\int_R K \, dA > 0$

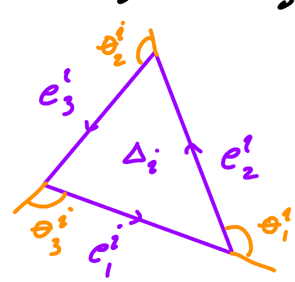
**Remark:** In $\mathbb{R}^3$ the surface below is a sphere but since it has two disjoint geodesics we know there must be non-positive curvature in



**Proof of Th$^{\underline{m}}$ 4 (Gauss-Bonnet):**

given the triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^{n}$ (can use finite number since $\Sigma$ compact)

let $e_j^i$ be arc length param of edges $\Delta_i$ and $\theta_j^i$ the exterior angles



apply Th$^{\underline{m}}$3 to each $\Delta_i$ to get

$$\sum_{j=1}^{3} \int_{e_j^i} k_g(s)\,ds + \int_{\Delta_i} K\,dA + \sum_{j=1}^{3} \theta_j^i = 2\pi$$

sum over all $\Delta_i$ to get

$$⊛ \quad \boxed{\sum_{i=1}^{n} \left[ \sum_{j=1}^{3} \int_{e_j^i} k_g(s)\,ds + \int_{\Delta_i} K\,dA + \sum_{j=1}^{3} \theta_j^i \right] = 2\pi n}$$

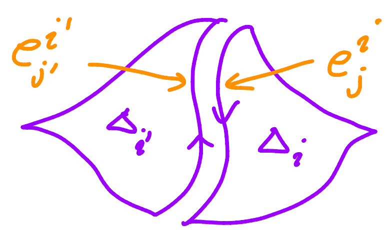
__Faces:__ since $\Sigma = \bigcup_{i=1}^{n} \Delta_i$ and $(\text{interior } \Delta_i) \cap (\text{interior } \Delta_j) = \emptyset$

we have $\int_\Sigma K\,dA = \int_{\cup \Delta_i} K\,dA = \sum_{i=1}^{n} \int_{\Delta_i} K\,dA$

so second term in ⊛ is $\int_\Sigma K\,dA$

__Edges:__ an edge of $\Delta_i$ can either be on $\partial\Sigma$ or not (i.e. interior)

if edge is interior, then there is some $\Delta_{i'}$ such that



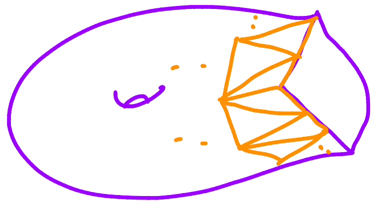

so $\int_{e_j^i} k_g(s)\,ds = -\int_{e_{j'}^{i'}} k_g(s)\,ds$

since parameterized in opposite direction

$\therefore$ in $\sum_{i=1}^{n} \left( \sum_{j=1}^{3} \int_{e_j^i} k_g(s)\,ds \right)$

all interior edges contribute 0

so only have contribution from $\partial\Sigma$ edges



<u>note:</u> sum of $\int_{e^i_j} \kappa_g(s)\,ds$ for these edges

is just $\displaystyle\sum_{i=1}^{h} \int_{C_i} \kappa_g(s)\,ds$

since $\bigcup C_i = \bigcup$ boundary edges

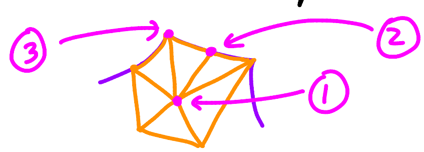
$\therefore$ first term in $(*)$ is $\displaystyle\sum_{i=1}^{h} \int_{C_i} \kappa_g(s)\,ds$

<u>Verticies:</u> 3 types of verticies

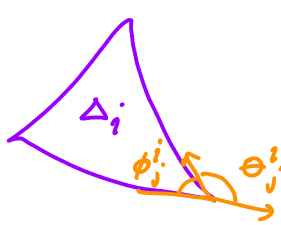
(1) $v$ on interior of $\Sigma$

(2) $v$ a regular point of $\partial\Sigma$

(3) $v$ a corner point of $\partial\Sigma$

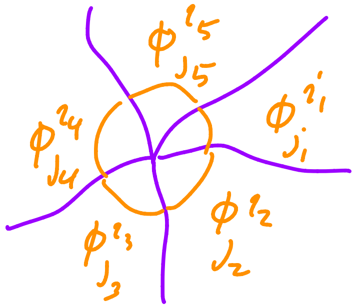



let $\phi^i_j$ be the <u>interior</u> angle at a corner of $\Delta_i$




$$\theta^i_j = \pi - \phi^i_j$$

the sum of the $\theta^i_j$ associated to an interior vertex is




$$\sum_{\substack{\text{around}\\\text{interior}\\\text{vertex}}} \theta^i_j = \sum \pi - \phi^i_j$$
$$= \pi(\#\,\text{triangles}) - \sum \phi^i_j$$
$$= \pi(\#\,\text{triangles}) - 2\pi$$
$$= \pi(\#\,\text{edges}) - 2\pi$$

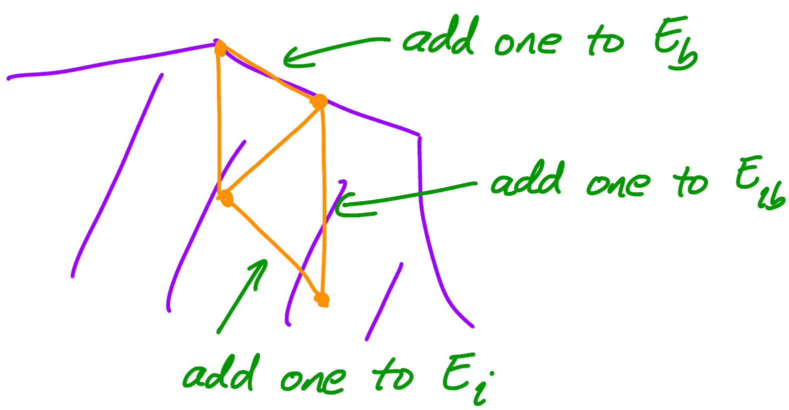
so if $V_i$ = # interior verticies

$V_b$ = # boundary verticies

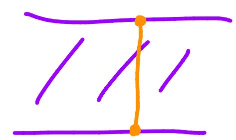
$E_i$ = # edges with both verticies on interior

$E_{ib}$ = # edges with one boundary and one interior vertex

$E_b$ = # edges with both verticies on boundary




we can assume no edges like

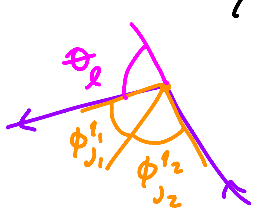


by subdividing triangles

then

$$\sum_{\substack{\text{interior}\\\text{verticies}}} \theta^i_j = \pi(\underbrace{2E_i}_{\substack{\text{each edge}\\\text{contributes } \pi\\\text{to 2 verticies}}} + \underbrace{E_{ib}}_{\substack{\text{contributes } \pi\\\text{to only one}\\\text{vertex}}}) - 2\pi V_i$$

at a corner boundary vertex

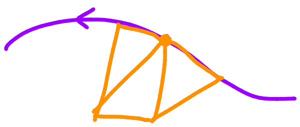


<u>note:</u> $\sum \theta_j^i$ at this vertex

$$= \sum (\pi - \phi_j^i)$$
$$= \pi(\#\text{ triangles}) - \sum \phi_j^i$$
$$= \pi[\#(\text{interior edges}) + 1] - (\pi - \theta_\ell)$$
$$= \pi(\#\text{ interior edges}) + \theta_\ell$$

similarly at a regular boundary vertex you get

$\sum \theta_j^i$ at this vertex



$$= \pi(\#\text{ interior edges})$$ (since $\theta_i$ is essentially 0)

so $\displaystyle\sum_{\substack{\text{boundary}\\ \text{vertices}}} \theta_j^i = \pi E_{ib} + \sum_{i=1}^{k} \theta_i$

now if $F = \#\text{ faces} = n$

$E = \#\text{ edges} = E_i + E_{ib} + E_b$

$V = \#\text{ vertices} = V_i + V_b$

then ⊛ becomes

$$\sum_{i=1}^{k} \int_{C_i} \kappa_g(s)\,ds + \int_\Sigma K\,dA + 2\pi(E_i + E_{ib} - V_i) + \sum_{i=1}^{k} \theta_i = 2\pi F$$

so $$\sum_{i=1}^{k} \int_{C_i} \kappa_g(s)\,ds + \int_\Sigma K\,dA + \sum_{i=1}^{k} \theta_i = 2\pi(F - E_i - E_{ib} + V_i)$$

$$= 2\pi(F - E_i - E_{ib} - E_b + V_b + V_i)$$

since $E_b = V_b$ $\longrightarrow$

$$= 2\pi(F - E + V) = 2\pi\chi(\Sigma)$$