C. Geodesic Coordinates

We now prove a claim used many times so far

Thm 10:

let Σ be a surface with Riemannian metric g any point $\vec{p} \in \Sigma$ is contained in a coordinate chart $\vec{f} : V \to \Sigma$

such that $g = \begin{pmatrix} g_{11}(u) & 0 \\ 0 & g_{22}(u,v) \end{pmatrix}$

moreover the curves $\vec{f}(u, v_0)$ are geodesics

To prove this theorem we need a result from ODE's

Fundamental Theorem of ODEs:

given $U \subset \mathbb{R}^n$ open set

$I \subset \mathbb{R}$ open interval containing 0

$\vec{x}_i \in U$ $\quad i = 0, \dots, k-1$ and

$H : U^k \times I \to \mathbb{R}^n$ a continuous function

that is Lipschitz in $\vec{x}$

(i.e. $\exists\, c$ such that

$$\| H(\vec{x},t) - H(\vec{y},t) \| \le c \| \vec{x} - \vec{y} \|$$

$\forall\, \vec{x}, \vec{y} \in U$ and $t \in I$)

note: H differentiable on compact set $\Rightarrow$ H Lipschitz

Then the differential equation

$$\frac{d^k}{dt^k} \vec{x} = H(\vec{x}, \vec{x}', \dots, \vec{x}^{(k-1)}, t)$$

$$\vec{x}(0) = \vec{x}_0$$

$$\vdots$$

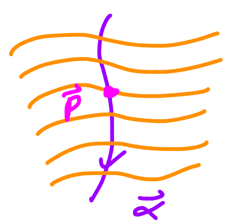
$$\vec{x}^{(k-1)}(0) = \vec{x}_{k-1}$$

has a unique solution $\vec{x}(t; \vec{x}_0, \ldots, \vec{x}_{k-1})$
defined on sum interval $I' \subset I$ containing 0
if H is differentiable then $\vec{x}(t; \vec{x}_0, \ldots, \vec{x}_{k-1})$ is differentiable in t and $\vec{x}_0, \ldots, \vec{x}_{k-1}$.

for a proof see any good book on ODE

Proof: let $\vec{\alpha}: (-\varepsilon, \varepsilon) \to \Sigma$ be any curve through $\vec{p}$ $(\vec{\alpha}(0) = \vec{p})$

Idea: build coords by taking geodesics perpendicular to $\vec{\alpha}$



let $\vec{x}_1(\vec{\alpha}(t)) = J(\vec{\alpha}'(t)) \dfrac{1}{\|\vec{\alpha}'(t)\|_g}$

where J = rotation by $\frac{\pi}{2}$ counter clockwise in $T_{\vec{\alpha}(t)}\Sigma$

note: $\vec{x}_1$ is a unit vector field along $\vec{\alpha}(t)$

recall the geodesic equations are

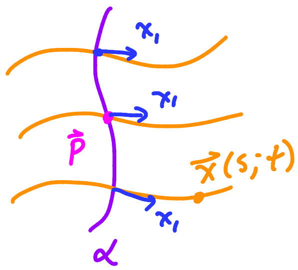
$$a''(s) = -\left(\Gamma_{11}^1 (a')^2 + 2\Gamma_{12}^1 a'b' + \Gamma_{22}^1 (b')^2\right)$$
$$b''(s) = \underbrace{-\left(\Gamma_{11}^2 (a')^2 + 2\Gamma_{12}^2 a'b' + \Gamma_{22}^2 (b')^2\right)}_{H((a,b),(a',b'),s)}$$

note use variable s to avoid confusion with t from $\vec{\alpha}$

so we can use Th$^{\underline{m}}$ on ODEs to get a sol$^{\underline{n}}$ to

$$\vec{x}''(s) = H(\vec{x}, \vec{x}', s)$$
$$\vec{x}(0) = \vec{\alpha}(t)$$
$$\vec{x}'(0) = \vec{x}_1(\vec{\alpha}(t))$$

call sol^{n} $\vec{x}(s;t)$



clearly $\vec{x}(s;t_0)$ is a geodesic $\forall t_0$

$$\vec{x}(0,0) = \vec{\alpha}(0) = \vec{p}$$

$$D\vec{x}_{(0,0)} = \left[\frac{\partial \vec{x}}{\partial s}(0,0) \quad \frac{\partial \vec{x}}{\partial t}(0,0) \right]$$
$$= \left[\vec{x}_1(0) \quad \vec{\alpha}'(0) \right]$$
$$= \left[\frac{J(\vec{\alpha}'(0))}{\|\vec{\alpha}'(0)\|} \quad \vec{\alpha}'(0) \right]$$

linearly independent
so $D\vec{x}_{(0,0)}$ rank 2

so $D\vec{x}_{\vec{q}}$ rank 2 for all $\vec{q}$ near $(0,0)$

Inverse function theorem says $\vec{x}$ is invertable near $\vec{x}(0,0)$

so $\vec{x}$ is injective near $(0,0)$

$\therefore$ $\vec{x}$ gives coordinates near $\vec{p}$

(rename $\vec{f}(u,v) = \vec{x}(u,v)$ for standard notation)

now consider
$$g_{12} = \vec{f}_u \cdot \vec{f}_v$$
$$\frac{\partial}{\partial u} g_{12} = \vec{f}_{uu} \cdot \vec{f}_v + \vec{f}_u \cdot \vec{f}_{vu}$$

but $\vec{f}_u \cdot \vec{f}_u = \frac{\partial \vec{x}}{\partial s} \cdot \frac{\partial \vec{x}}{\partial s} = 1$

so $\frac{\partial}{\partial v} \vec{f}_u \cdot \vec{f}_u = 2\vec{f}_{uv} \cdot \vec{f}_u = 0$

$\therefore \frac{\partial}{\partial u} g_{12} = \vec{f}_{uu} \cdot \vec{f}_v$

now ↙ tangential component

$$(\vec{f}_{uu})^T = \left(\frac{\partial}{\partial u}(\vec{f}_u)\right)^T = \left(\frac{\partial}{\partial s}\left(\frac{\partial \vec{x}}{\partial s}\right)\right)^T = \nabla_{\frac{\partial \vec{x}}{\partial s}} \frac{\partial \vec{x}}{\partial s} = 0$$

↑ $\vec{x}(s,t_0)$ a geodesic

so $\vec{f}_{uu} \cdot \vec{f}_v = 0$

and $\frac{\partial g_{12}}{\partial u} = 0$

thus $g_{12}(u,v) = g_{12}(0,v) = \frac{\partial \vec{x}}{\partial s}(0,v) \cdot \frac{\partial \vec{x}}{\partial t}(0,v)$

$$= \frac{J \vec{\alpha}'(v)}{\|\vec{\alpha}'(v)\|} \cdot \vec{\alpha}'(v) = 0$$

the fact g_{11} only depends on v was shown in proof of Theorem V.10 

note we showed more!

Cor 11:

let Σ be a surface with Riemannian metric g any point $\vec{p} \in \Sigma$ is contained in a coordinate chart with

$$g = \begin{pmatrix} 1 & 0 \\ 0 & g_{22}(u,v) \end{pmatrix}$$

Remark: Th$^{\underline{m}}$ V.1 gives

$$K = -\frac{1}{2\sqrt{g_{22}}} \left(\frac{(g_{22})_u}{\sqrt{g_{22}}}\right)_u$$

$$= -\frac{1}{\sqrt{g_{22}}} \frac{d^2}{du^2} \sqrt{g_{22}}$$

Recall: given $\vec{v} \in T_{\vec{p}}(\Sigma)$

$\exists!$ geodesic $\vec{\gamma} : (-\varepsilon, \varepsilon) \to \Sigma$ such that

$\vec{\gamma}(0) = \vec{p}$

$\vec{\gamma}'(0) = \vec{v}$

denote it $\vec{\gamma}(t; \vec{v})$

lemma 12:

> if $\vec{\gamma}(t; \vec{v})$ is defined on $(-\varepsilon, \varepsilon)$
>
> then $\vec{\gamma}(t; \lambda\vec{v})$, $\lambda \neq 0$, is defined on $(-\varepsilon/\lambda, \varepsilon/\lambda)$
>
> and is given by $\vec{\gamma}(t; \lambda\vec{v}) = \vec{\gamma}(\lambda t; \vec{v})$

Proof: let $\vec{\alpha}(t) = \vec{\gamma}(\lambda t; \vec{v})$

$\vec{\alpha} : (-\varepsilon/\lambda, \varepsilon/\lambda) \to \Sigma$ is a parameterized curve

$\vec{\alpha}(0) = \vec{p}$

$\vec{\alpha}'(0) = \lambda \vec{\gamma}'(0; \vec{v}) = \lambda \vec{v}$

$\nabla_{\vec{\alpha}'(t)} \vec{\alpha}'(t) = \nabla_{\lambda\gamma'(0;\vec{v})} \lambda \gamma'(0; \vec{v}) = \lambda^2 \nabla_{\gamma'(t;v)} \gamma'(t;v) = 0$

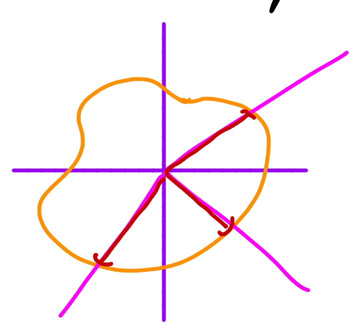
so $\vec{\alpha}$ is a geodesic and by uniqueness in Th$^{\underline{m}}$ V.9

we have $\vec{\gamma}(\lambda t; \vec{v}) = \vec{\alpha}(t) = \vec{\gamma}(t; \lambda\vec{v})$ ▦

now let U be the set of vectors $\vec{v}$ in $T_{\vec{p}}\Sigma$ such that

$\vec{\gamma}(1; \vec{v})$ is defined

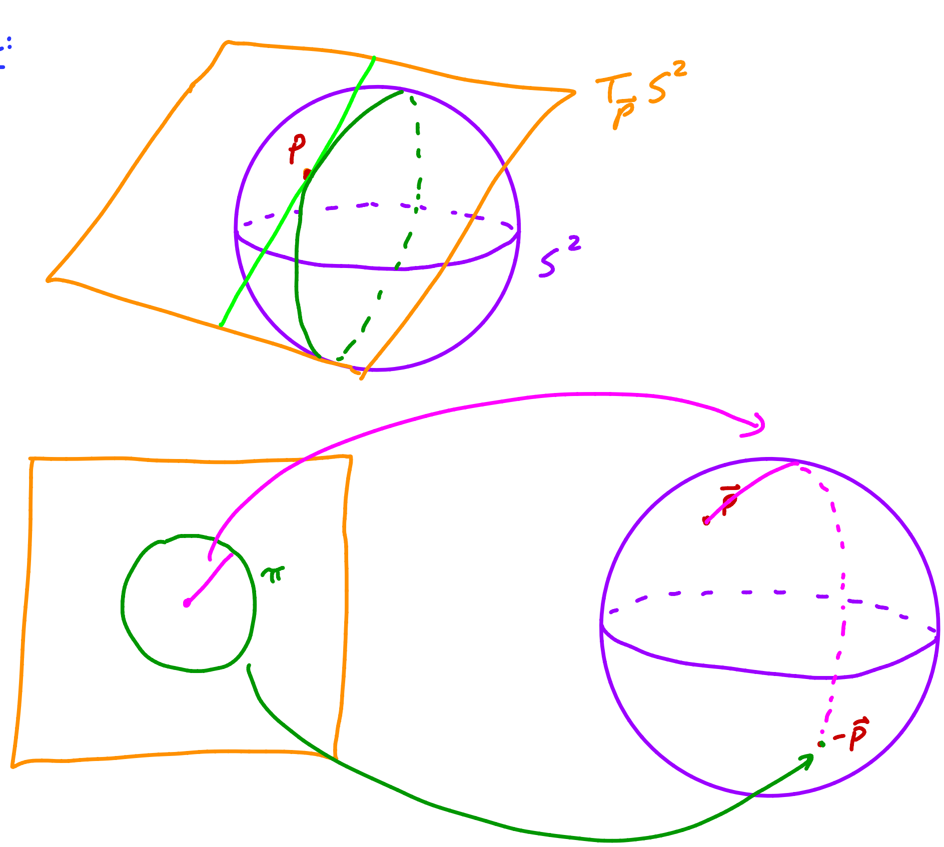
<u>note</u>: lemma says U is star shaped



the *exponential map* is defined as

$$\begin{aligned} \exp_{\vec{p}} : U \subseteq T_{\vec{p}}\Sigma &\longrightarrow \Sigma \\ \vec{v} &\longmapsto \begin{cases} \vec{\gamma}(1;\vec{v}) & \vec{v} \neq 0 \\ \vec{p} & \vec{v} = 0 \end{cases} \end{aligned}$$

example:



lemma 13:

> for all $\vec{p} \in \Sigma$ there is some $\varepsilon > 0$ such that $\exp_{\vec{p}}$ is defined and differentiable on a ball of radius ε about 0 in $T_{\vec{p}}\Sigma$

Proof: let C be the unit circle in $T_{\vec{p}}\Sigma$

fix some $R >> 0$

if $\vec{v} \in C$ let $\varepsilon(\vec{v})$ be the largest $\varepsilon < R$ such that $\vec{\gamma}(t;\vec{v})$ is defined on $(-\varepsilon, \varepsilon)$

$Th^{\underline{m}}$ V.9 $\Rightarrow \varepsilon(\vec{v}) > 0 \quad \forall \vec{v}$

Fundamental $Th^{\underline{m}}$ of ODE $\Rightarrow \varepsilon : C \to \mathbb{R}$ is continuous

since C is compact, ε takes on its global min ε_0

let B = Ball of radius ε_0 in $T_{\vec{p}}\Sigma$

$\vec{v} \in B, \vec{v} \neq 0 \Rightarrow \frac{\vec{v}}{\|\vec{v}\|} \in C$ so $\vec{\gamma}\left(t; \frac{\vec{v}}{\|\vec{v}\|}\right)$ defined on $(-\varepsilon_0, \varepsilon_0)$

and $\|\vec{v}\| < \varepsilon_0$ so $\vec{\gamma}\left(\|\vec{v}\|; \frac{\vec{v}}{\|\vec{v}\|}\right) = \vec{\gamma}(1; \vec{v})$ is defined

so $exp_{\vec{p}}(\vec{v})$ well-defined

$exp_{\vec{p}}(\vec{v})$ is differentiable by the Fundamental Theorem of ODE

Lemma 14:

> for $\vec{p} \in \Sigma$, there is some $\varepsilon > 0$ such that
>
> $$exp_{\vec{p}} : B_\varepsilon \to \Sigma$$
>
> ball of radius ε in $T_p\Sigma$
>
> is injective and
>
> $$D(exp_{\vec{p}})_{\vec{q}}$$
>
> is rake 2 for all $\vec{q} \in B_\varepsilon$

Proof: consider $\vec{\gamma}(t; \vec{v}) = \vec{\gamma}(1; t\vec{v}) = exp_{\vec{p}}(t\vec{v}) = exp_{\vec{p}} \circ \vec{\alpha}(t)$

where $\vec{\alpha}(t) = t\vec{v}$

$Th^{\underline{m}}$ III.1 says

$$D(exp_{\vec{p}})_{(0,0)}(\vec{v}) = \frac{d}{dt}\left(exp_{\vec{p}} \circ \vec{\alpha}(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\, \vec{\gamma}(t; \vec{v})\Big|_{t=0} = \vec{v}$$

so $D(exp_{\vec{p}})_{(0,0)}(\vec{v}) = \vec{v}$

i.e. $D(exp_{\vec{p}}) = T_{\vec{p}}\Sigma \to T_{\vec{p}}\Sigma$ is the identity map

$\therefore$ rank 2

The Inverse Function Theorem says $exp_{\vec{p}}$ is invertable near $(0,0)$

so it is injective $\boxplus$

note: The lemma says

$$\exp_{\vec{p}} : B_\varepsilon \to \Sigma$$

is a coordinate chart about $\vec{p}$

if we fix $\{\vec{w}_1, \vec{w}_2\}$ an orthonormal basis for $T_{\vec{p}}\Sigma$, then

$$\vec{f}(u,v) = \exp_{\vec{p}}(u\vec{w}_1 + v\vec{w}_2)$$

is called a normal coordinate chart

lemma 15:

In normal coordinates we have

1) $g(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) first derivatives of $\{g_{ij}(u,v)\}$ at $(0,0)$ vanish

3) $\Gamma_{ij}^k(0,0) = 0$

Proof: $\vec{f}_u(0,0) = \frac{d}{du} \exp_{\vec{p}}(u\vec{w}_1)\Big|_{u=0} = \vec{w}_1$

$\vec{f}_w(0,0) = \vec{w}_2$

so $g(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $\vec{w}_1, \vec{w}_2$ orthonormal

note: $\vec{\gamma}(t) = \exp_{\vec{p}}(t(r\vec{w}_1 + s\vec{w}_2))$ is a geodesic for fixed r,s

i.e. if $a(t) = rt$ and $b(t) = st$

then $\vec{f}(a(t), b(t))$ is a geodesic

$\vec{\gamma}$ a geodesic iff

$$(1) \quad a'' + \Gamma^1_{11}(a')^2 + 2\Gamma^1_{12} a'b' + \Gamma^1_{22}(b')^2 = 0$$

$$(2) \quad b'' + \Gamma^2_{11}(a')^2 + 2\Gamma^2_{12} a'b' + \Gamma^2_{22}(b')^2 = 0$$

but $a' = r \qquad a'' = 0$

$\quad\;\; b' = s \qquad b'' = 0$

$\forall r,s$, $\vec{\gamma}(0) = \vec{p}$ so we get for $s=0$ and $r=1$, (1) becomes

$$\Gamma^1_{11} = 0$$

so (1) is $2\Gamma^1_{12} rs + \Gamma^1_{22} s^2 = 0$

now $r=0$, $s=1 \Rightarrow \Gamma^1_{22} = 0$

and $r = 1 = s \Rightarrow \Gamma^1_{12} = 0$

similarly $\Gamma^2_{11} = \Gamma^2_{22} = \Gamma^2_{12} = 0$ using (2)

finally

lemma V.2

$$0 = \Gamma^1_{11} = (g_{11})_u + (g_{11})_u - (g_{11})_u$$

so $(g_{11})_u = 0$

$$0 = \Gamma^1_{12} = (g_{21})_u + (g_{11})_v - (g_{12})_u$$

so $(g_{11})_v = 0$

similarly for other g_{ij}

Using $\exp_{\vec{p}} : \overset{\subset T_{\vec{p}}\Sigma = \mathbb{R}^2}{B_\varepsilon} \to \Sigma$ we also get geodesic polar coordinates by taking polar coords (r,θ) on $\mathbb{R}^2$ and defining

$$\vec{f}(r,\theta) = \exp_{\vec{p}}\left((r\cos\theta)\vec{w}_1 + (r\sin\theta)\vec{w}_2\right)$$

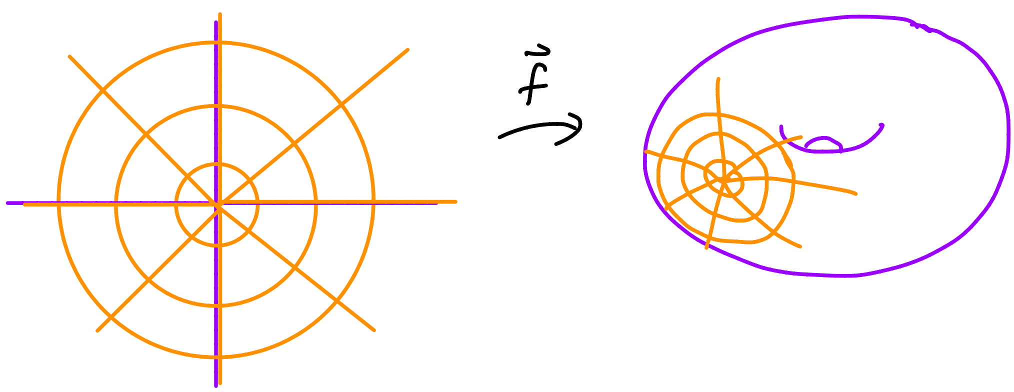
where $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal basis for $T_{\vec{p}}\Sigma$

we call the curve C_R parameterized by

$$\vec{\gamma}(t) = \exp_{\vec{p}}\left((R\cos t)\vec{w}_1 + (R\sin t)\vec{w}_2\right) \qquad t \in [0, 2\pi]$$

the <u>geodesic circle of radius R</u>

(note C_R is not a geodesic and might not be a circle!)



lemma 16:

In geodesic polar coordinates (away from $r=0$) we have

$$g = \begin{pmatrix} 1 & 0 \\ 0 & G(r,\theta) \end{pmatrix}$$

where

$$\sqrt{G(r,\theta)} = r - \frac{1}{6}K(\vec{p})r^3 + R(r,\theta)$$

and

$$\lim_{r\to 0} \frac{R(r,\theta)}{r^3} = 0$$

(i.e. up to order 3 $\sqrt{G} = r - \frac{1}{6}K(\vec{p})r^3$)

Proof: Proof of lemma 14 says that

$$\vec{\gamma}(t) = \exp_{\vec{p}}(t\vec{v}) \quad \text{is a geodesic}$$

so $\vec{\gamma}(r) = \vec{f}(r, \theta_0)$ is a geodesic

and $\vec{f}_r = \vec{\gamma}\,'(r) = ((\cos\theta)\vec{v}_1 + (\sin\theta)\vec{v}_2)$

has unit length

so $\vec{f}_r \cdot \vec{f}_r = 1$

since $\vec{f}(a(t), b(t))$ for $a(t) = t$, $b(t) = \theta_0$

is a geodesic, the geodesic equations give

$$a''(t) + \Gamma^1_{11}(a')^2 + 2\Gamma^1_{12}\, a'b' + \Gamma^1_{22}(b')^2 = 0$$

$$b''(t) + \Gamma^2_{11}(a')^2 + 2\Gamma^2_{12}\, a'b' + \Gamma^2_{22}(b')^2 = 0$$

so $\Gamma^2_{11} = 0$

$$0 = \Gamma^2_{11} = g^{21}\,\tfrac{1}{2}\cancelto{0}{\left((g_{11})_r + (g_{11})_r - (g_{11})_r\right)}$$

$$+\; g^{22}\,\tfrac{1}{2}\left((g_{12})_r + (g_{12})_r - \cancelto{0}{(g_{11})_\theta}\right)$$

we know $g^{22} \neq 0$ (if $r \neq 0$) so

$$(g_{12})_r = 0$$

and g_{12} only depends on θ

let $\vec{\alpha}_{r_0}(\theta) = \vec{f}(r_0, \theta)$ so

$$g_{12}(r, \theta) = \vec{f}_r \cdot \vec{f}_\theta = \vec{\gamma}_{\theta_0}'(r_0) \cdot \vec{\alpha}_{r_0}'(\theta_0)$$

note $\vec{\alpha}_0(\theta) = \vec{p}$ $\forall\theta$ so $\vec{\alpha}_0\,'(\theta) = 0$

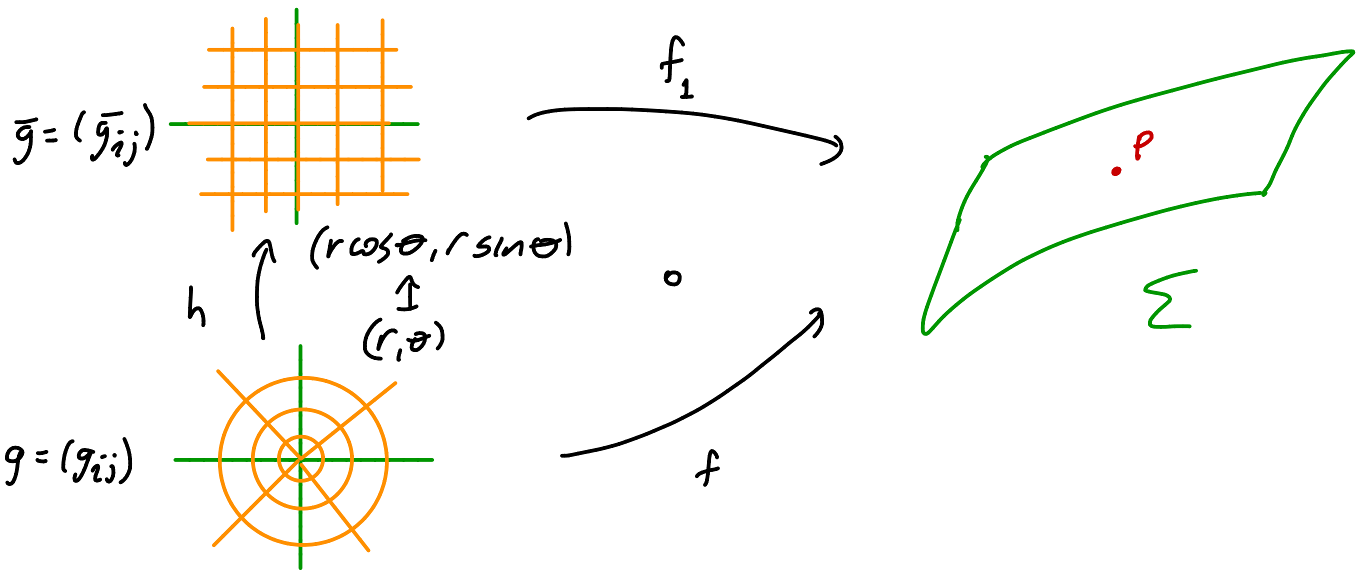
thus $\displaystyle\lim_{r_0 \to 0} g_{12}(r_0, \theta_0) = \lim_{r_0 \to 0} \vec{\gamma}_{\theta_0}'(r_0) \cdot \vec{\alpha}_{r_0}(\theta_0) = 0$

so $g_{12} = 0$

and $$g = \begin{pmatrix} 1 & 0 \\ 0 & G(r, \theta) \end{pmatrix}$$

now consider the Taylor expansion of $\sqrt{G}$

note from above we have $\sqrt{G(0,\theta)} = 0$



from lemma 15 $\quad (\bar{g}_{ij}(0,0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>note</u>:
$$\vec{v}^{\,t} g\, \vec{w} = g(\vec{v},\vec{w}) = \bar{g}\,(Dh(\vec{v}), Dh(\vec{w}))$$
$$= (Dh(\vec{v}))^T \bar{g}\, (Dh(\vec{w}))$$
$$= \vec{v}^{\,T}\, Dh^T \bar{g}\, Dh\, \vec{w}$$

so $g = Dh^T \bar{g}\, Dh$

and $\quad Dh = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$$\sqrt{G} = \sqrt{g_{22}} = \sqrt{\det g} = \sqrt{\det Dh^T \det \bar{g} \det Dh} = r\sqrt{\det \bar{g}}$$

so $\displaystyle \lim_{r\to 0} \frac{\partial}{\partial r}\sqrt{G} = \lim_{r\to 0}\left(\sqrt{\det\bar{g}} + r\frac{\partial}{\partial r}\sqrt{\det\bar{g}}\right)$

$$= \sqrt{\det \bar{g}(0,0)} = 1$$

and $\lim_{r\to 0} \frac{\partial^2}{\partial r^2}\sqrt{G} = \lim_{r\to 0}\left(\frac{\partial}{\partial r}\sqrt{\det \bar{g}} + \frac{\partial}{\partial r}\sqrt{\det\bar{g}} + r\frac{\partial^2}{\partial r^2}\sqrt{\det\bar{g}}\right)$

$$= \lim_{r\to 0} 2\frac{\partial}{\partial r}\sqrt{\det\bar{g}} = 0$$

this involves 1^{st} derivs of $\bar{g}_{ij}$ at $(0,0)$ by lemma 15 these $=0$

by the remark after Cor 11 we have

$$K(r,\theta) = -\frac{1}{\sqrt{G(r,\theta)}}\left(\sqrt{G(r,\theta)}\right)_{rr} \quad \text{for } r \neq 0$$

so $(\sqrt{G})_{rr} = -\sqrt{G(r,\theta)}\, K(r,\theta)$

(note: also $\Rightarrow \lim_{r\to 0}(\sqrt{G})_{rr} = 0$)

and $(\sqrt{G})_{rrr} = -\left(\sqrt{G(r,\theta)}\right)_r K(r,\theta) - \sqrt{G(r,\theta)}\left(K(r,\theta)\right)_r$

$\therefore \lim_{r\to 0}(\sqrt{G})_{rrr} = -K(\vec{p})$

Thus Taylor's th^{m} gives

$$\sqrt{G} = 0 + 1r + \tfrac{1}{2}0r^2 + \tfrac{1}{6}\left(-K(\vec{p})\right)r^3 + \text{h.o.t.}$$

$$= r - \tfrac{1}{6}K(\vec{p})r^3 + \text{h.o.t.}$$

Th^{m} 17:

let Σ be a surface with Riemannian metric g

recall for $\vec{p}\in\Sigma$ the geodesic circle of radius R

is $C_R = \{\exp_{\vec{p}}((R\cos\theta)\vec{w}_1 + (R\sin\theta)\vec{w}_2) : \theta\in[0,2\pi]\}$

where $\vec{w}_1, \vec{w}_2$ orthonormal basis for $T_{\vec{p}}\Sigma$

let $L(R) = $ length C_R and

$A(R) = $ area of disk C_R bounds

Then the Gauss curvature is

$$K(\vec{p}) = \lim_{R\to 0} \frac{3}{\pi}\left(\frac{2\pi R - L(R)}{R^3}\right)$$

$$K(\vec{p}) = \lim_{R\to 0} \frac{12}{\pi}\left(\frac{\pi R^2 - A(R)}{R^4}\right)$$

Remark: So if $K(\vec{p}) > 0$, then circles of small radius about $\vec{p}$ are shorter and enclose less area than Euclidean circles

and similar comments apply to $K(\vec{p}) < 0$

Proof: Parameterize C_R by

$$\vec{\alpha}_R(t) = \exp_{\vec{p}}\left((R\cos t)\vec{w}_1 + (R\sin t)\vec{w}_2\right)$$
$$= \vec{f}(R, t)$$

geodesic polar coords

so $\vec{\alpha}_R' = \vec{f}_\theta(R,t)$

lemma 16

and $\|\vec{\alpha}_R'\| = \sqrt{f_\theta \cdot f_\theta} = \sqrt{G} = R - \frac{1}{6}K(\vec{p})R^3 + \text{h.o.t}$

so $$L(R) = \int_0^{2\pi} \|\vec{\alpha}_R'(t)\|_g \, dt = \int_0^{2\pi} \left(R - \tfrac{1}{6}K(\vec{p})R^3 + \text{h.o.t.}\right)dt$$

$$= \left(R - \tfrac{1}{6}K(\vec{p})R^3\right)t\Big|_0^{2\pi} + \text{h.o.t in } R$$

$$= 2\pi R - \tfrac{1}{6}K(\vec{p})R^3\, 2\pi + \text{h.o.t. in } R$$

$$\therefore \quad K(\vec{p}) = \frac{3}{\pi}\,\frac{2\pi R - L(R)}{R^3} + \underbrace{\cancel{\frac{\text{h.o.t. in } R}{R^3}}}_{\to 0 \text{ as } R\to 0}$$

so $$K(\vec{p}) = \lim_{R\to 0} \frac{3}{\pi}\left(\frac{2\pi R - L(R)}{R^3}\right)$$

exercise: compute the area of the disk C_R bounds and establish the other formula

Th$^{\underline{m}}$ 18:

If (Σ, g) and (Σ', g') are two surfaces with Riem. metrics both having constant curvature K and $\vec{p} \in \Sigma$ and $\vec{q} \in \Sigma'$

Then there are neighborhoods V of $\vec{p}$ in Σ and V' of $\vec{p}'$ in Σ' and an isometry $\phi : V \to V'$ taking $\vec{p}$ to $\vec{p}'$

(This says any two Riemannian surfaces with the same constant Gauss curvature are locally isometric)

Proof: recall in geodesic polar coordinates

$$K = -\frac{1}{\sqrt{g_{22}}}(\sqrt{g_{22}})_{rr}$$

so $\boxed{(\sqrt{g_{22}})_{rr} + K\sqrt{g_{22}} = 0}$

Case 1: $K = 0$

so $(\sqrt{g_{22}})_{rr} = 0$

so $(\sqrt{g_{22}})_r = g(\theta)$ (doesn't depend on r)

by lemma 16 we know

$$\lim_{r \to 0} (\sqrt{g_{22}})_r = 1$$

so $g(\theta) = 1$ for all θ and so $(\sqrt{g_{22}})_r = 1$

thus integrating we find $\sqrt{g_{22}} = r + f(\theta)$

but since $\lim_{r \to 0} \sqrt{g_{22}} = 0$ we see $f(\theta) = 0$

lemma 16

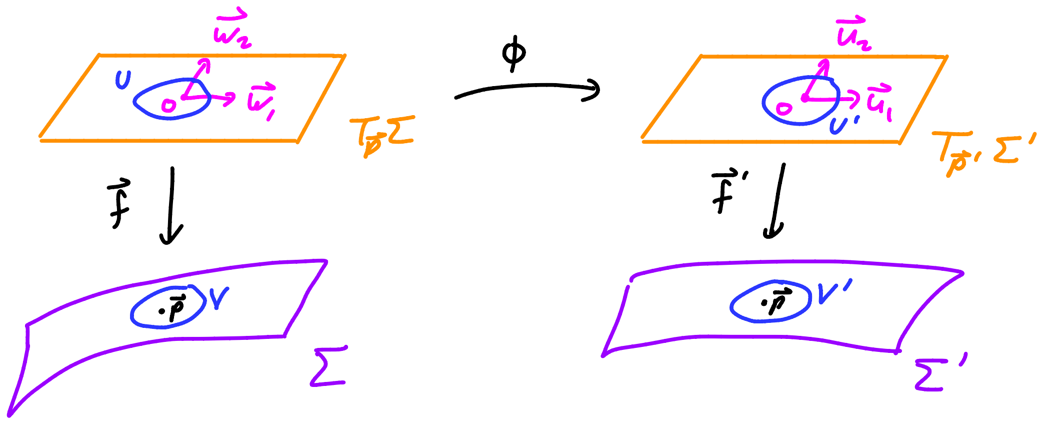
so the metric in geodesic normal coords is

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

now given $\vec{p} \in \Sigma$ and $\vec{p}\,' \in \Sigma'$

choose an orthonormal basis $\vec{w}_1, \vec{w}_2$ for $T_{\vec{p}}\Sigma$

" " $\vec{u}_1, \vec{u}_2$ for $T_{\vec{p}'}\Sigma'$



let ϕ be the linear map $T_{\vec{p}}\Sigma \to T_{\vec{p}'}\Sigma'$

sending $\vec{w}_1 \mapsto \vec{u}_1$

$\vec{w}_2 \mapsto \vec{u}_2$

let $\vec{f}(r,\theta) = \exp_{\vec{p}}\left((r\cos\theta)\vec{w}_1 + (r\sin\theta)\vec{w}_2\right)$

be geodesic polar coords on Σ near $\vec{p}$ on the set $U \subset T_{\vec{p}}\Sigma$ to $V \subset \Sigma$

similarly for $\vec{f}\,'(r',\theta') = \exp_{\vec{p}'}\left((r'\cos\theta')\vec{u}_1 + (r'\sin\theta')\vec{u}_2\right)$

are geodesic polar coords on Σ' near $\vec{p}\,'$ on the set $U' \subset T_{\vec{p}'}\Sigma'$ to $V' \subset \Sigma'$

define $\psi : V \to V'$ by

$$\psi(\vec{q}) = \vec{f}\,' \circ \phi \circ \vec{f}^{-1}(\vec{q})$$

exercise: this is an isometry taking

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \text{ in } (r,\theta) \text{ coords on } \Sigma$$

$$\text{to } \begin{pmatrix} 1 & 0 \\ 0 & (r')^2 \end{pmatrix} \text{ in } (r',\theta') \text{ coords on } \Sigma'$$

Case 2: $K > 0$

now $(\sqrt{g_{22}})_{rr} + K\sqrt{g_{22}} = 0$

has sol$^{\underline{n}}$ $\sqrt{g_{22}} = A(\theta)\cos\sqrt{K}\,r + B(\theta)\sin\sqrt{K}\,r$

since $\lim_{r\to 0}\sqrt{g_{22}} = 0$ we see $A(\theta) = 0$

and since $(\sqrt{g_{22}})_r = B(\theta)\sqrt{K}\cos\sqrt{K}\,r$

and $\lim_{r\to 0}(\sqrt{g_{22}})_r = 1$ we see $B(\theta) = \frac{1}{\sqrt{K}}$

so

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{K}\sin^2\sqrt{K}\,r \end{pmatrix}$$

argument now same as in Case 1

Case 3: $K < 0$

we get $\sqrt{g_{22}} = A(\theta)\cosh\sqrt{-K}\,r + B(\theta)\sinh\sqrt{-K}\,r$

and arguing as above we get

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{K}\sinh^2\sqrt{-K}\,r \end{pmatrix}$$

and now finish as in Case 1