D. Prescribing Curvature

What are the possible Gauss curvatures for a fixed surface?

Given a surface Σ, fix a Riemannian metric g

the simplest way to get a new metric is to multiply g by a positive function

this is called a conformal change of metric

(note that lengths change but angles do not since

$$\frac{g(\vec{u},\vec{v})}{\sqrt{g(\vec{u},\vec{u}),g(\vec{v},\vec{v})}} = \cos\theta = \frac{fg(\vec{u},\vec{v})}{\sqrt{fg(\vec{u},\vec{u})fg(\vec{v},\vec{v})}}$$

so we will consider new metrics

$$g' = e^{2f} g$$

where f is any function (the 2 is for convenience)

exercise: if $K(\vec{p})$ is the Gauss curvature of g and $K'(\vec{p})$ is the Gauss curvature of $g' = e^{2f} g$

then $$K'(\vec{p}) = e^{-2f(\vec{p})}(-\Delta f(\vec{p}) + K(\vec{p}))$$

where $$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

so given g (and hence K)

to find g' with curvature K' we need to solve

$$\boxed{\Delta f(\vec{p}) = K(\vec{p}) - e^{2f(\vec{p})} K'(\vec{p})} \quad (*)$$

for f

let's try to do this for K' constant (i.e. does every surface have a metric with constant Gauss curvature?)

P.D.E. Fact:

given $K(\vec{p})$

$\exists$ a unique function γ and constant $\tilde{K}$ s.t.

1) $\Delta \gamma(\vec{p}) = K(\vec{p}) - \tilde{K}$ and

2) $\int_\Sigma \gamma \, dA = 0$

so equation $(*)$ can be written

$$\Delta f = \tilde{K} + \Delta \gamma - e^{2f} K'$$

or we want to solve

$$\boxed{\Delta \psi - \tilde{K} = -K' e^{2(\psi + \gamma)}} \quad (**)$$

where $\psi = f - \gamma$

Case 1: $\Sigma = T^2$

Gauss-Bonnet says $\int_\Sigma K \, dA = 0$

so
$$
\begin{aligned}
0 &= \int_\Sigma K \, dA \\
&= \int_\Sigma (\tilde{K} + \Delta\gamma) \, dA \\
&= \tilde{K} \int_\Sigma dA + \int_\Sigma \Delta\gamma \, dA
\end{aligned}
$$

constant

"Green's Th$^{\underline{m}}$" says

$$\int_\Sigma \Delta\gamma \, dA = \int_{\partial A} \frac{\partial\gamma}{\partial x} dy - \frac{\partial\gamma}{\partial y} dx$$

$\emptyset$

$= 0$

exercise: use local coordinates and a triangulation of Σ to make this rigorous

$$= \tilde{K} \, \mathrm{area}(\Sigma)$$

so we have $\tilde{K} = 0$

and we know K' must be zero too

so (**) becomes

$$\Delta \varphi = 0$$

and $\varphi = 0$ is a solution!

so if we take $f = \gamma$ then

$g' = e^{2\gamma} g$ will have constant Gauss curvature $K' = 0$

<u>Case 2:</u> $\chi(\Sigma) < 0$

as above we must have $\tilde{K}$ and $K' < 0$

let $\mathcal{F} = \{$ functions $\psi : \Sigma \to \mathbb{R}$ such that $\int_\Sigma e^{2(\psi + \gamma)} dA = 1 \}$

consider
$$I : \mathcal{F} \to \mathbb{R}$$
$$\psi \mapsto \frac{1}{2} \int_\Sigma (\|\nabla\psi\|^2 + \tilde{K}\psi)\, dA$$

here $\nabla\psi$ is the gradient of ψ defined as the unique vector field such that

$g(\nabla\psi, \vec{v}) = \psi_{\vec{v}}$ ← directional derivative

for all $\vec{v}$

<u>exercise:</u> Show $\nabla\psi$ exists and

for functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

we have
$$\nabla\psi = \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{bmatrix}$$

let $\nu = \inf_{\psi \in \mathcal{F}} I(\psi)$

Claim 1: ν is finite

to see this note

$$\int_\Sigma 2(\psi+\gamma)\,dA = \int_\Sigma \log\left(e^{2(\psi+\gamma)}\right) dA$$

$$\leq \log \underbrace{\int_\Sigma e^{2(\psi+\gamma)}\,dA}_{=1}$$

Show: $\int \log h = \log \int h$

$$= 0$$

so $\int_\Sigma \psi\,dA \leq -\int_\Sigma \gamma\,dA = 0$

thus $I(\psi) = \frac{1}{2}\left(\int_\Sigma \|\nabla\psi\|^2 + \underbrace{\underset{\leq 0}{\tilde{K}} \underset{\leq 0}{\int \psi\,dA}}_{\geq 0}\right)$

$$\geq 0$$

so $\inf I \geq 0$ ✓

Claim 2: $\exists\, \psi \in \mathcal{F}$ such that $I(\psi) = \nu$

Idea: take a sequence ψ_i such that

$$I(\psi_i) \to \nu$$

then use functional analysis to show

$$\psi_i \to \psi \quad \text{and} \quad I(\psi) = \nu$$

Claim 3: ψ is a solution to (**)

(and so $f = \psi + \gamma$ is a solution to (*)

and $g' = e^{2f} g$ has Gauss curvature K')

note: ψ is a minimum of I on $\mathcal{F}$ so it is a critical point

recall: Lagrange multipliers

$f: \mathbb{R}^n \to \mathbb{R}$ and $\vec{x}$ is a critical point of f restricted to h = constant

then there is a constant c such that

$$\nabla f(\vec{x}) = c \nabla h(\vec{x})$$

thus $\quad Df_{\vec{x}}(\vec{v}) = g(\nabla f(\vec{x}), \vec{v}) = c\, g(\nabla h(\vec{x}), \vec{v}) = c\, Dh_{\vec{x}} \vec{v}$

ie. $Df_{\vec{x}}(\vec{v}) = c\, Dh_{\vec{x}}(\vec{v})$ for all $\vec{v}$

similarly $\; DI_\psi(\phi) = c\, DJ_\psi(\phi)$

where $J: \left\{ \begin{matrix} \text{functions} \\ \text{on } \Sigma \end{matrix} \right\} \to \mathbb{R}$

$$\phi \longmapsto \int_\Sigma e^{2(\phi - \gamma)} \, dA$$

now $\; DI_\psi(\phi) = \frac{d}{dt} I(\psi + t\phi) \Big|_{t=0}$

$$= \frac{d}{dt} \frac{1}{2} \int_\Sigma \left(\nabla(\psi + t\phi) \cdot \nabla(\psi + t\phi) + \tilde{K}(\psi + t\phi) \right) dA \Big|_{t=0}$$

$$= \frac{d}{dt} \int \left(\frac{1}{2} \nabla\psi \cdot \nabla\psi + t(\nabla\psi \cdot \nabla\phi) + t^2 \nabla\phi \cdot \nabla\phi + \tilde{K}(\psi + t\phi) \right) dA \Big|_{t=0}$$

$$= \int \nabla\psi \cdot \nabla\phi + \tilde{K}\phi \, dA$$

and

$$DJ_\psi(\phi) = \frac{d}{dt} J(\psi + t\phi) \Big|_{t=0}$$

$$= \frac{d}{dt} \int_\Sigma e^{2(\psi + t\phi + \gamma)} dA \Big|_{t=0}$$

$$= \int_\Sigma e^{2(\psi + t\phi + \gamma)} 2\phi \, dA \Big|_{t=0}$$

$$= \int_\Sigma 2\phi e^{2(\psi + \gamma)} dA$$

we know $\exists$ some c s.t.

$$(\mp) \qquad \underset{\displaystyle \int_\Sigma \nabla\psi \cdot \nabla\phi + \tilde{K}\phi \, dA}{\underset{\shortparallel}{DI_\psi(\phi)}} = \underset{\displaystyle c\int_\Sigma 2\phi e^{2(\psi+\gamma)} dA}{\underset{\shortparallel}{c\, Dh_\psi(\phi)}} \qquad \forall \phi$$

take $\phi = 1$ to get

$$\int_\Sigma \tilde{K}\, dA = c\,2\int_\Sigma \underbrace{e^{2(\psi+\gamma)}\, dA}_{=1} = 2c$$

$\parallel$ ← from PDE Fact above

$$\int_\Sigma K + \Delta\gamma\, dA$$

$$\parallel$$

$$\int_\Sigma K\, dA = 2\pi\chi(\Sigma)$$

↑ Gauss Bonnet!

so $(\dagger)$ becomes

$$\int_\Sigma \nabla\psi\cdot\nabla\phi + \tilde{K}\phi - 2\pi\chi(\Sigma)\,\phi\, e^{2(\psi+\gamma)}\, dA = 0$$

$\parallel$ Integrate by parts

$$\int_\Sigma \left(-\Delta\psi + \tilde{K} - 2\pi\chi(\Sigma)e^{2(\psi+\gamma)}\right)\phi\, dA$$

true for all ϕ

exercise: if $f: U \to \mathbb{R}$ is continuous, U open set in $\mathbb{R}^n$, and $\int_U fg = 0\ \forall g$ then $f=0$

Hint: if $f(x) \neq 0$, then $\exists$ nbhd V of x in U s.t. $f(y) \neq 0\ \forall y \in V$

let $g = \begin{cases} 1 & \text{near } x \\ 0 & \text{outside } V \\ \geq 0 & \text{everywhere} \end{cases}$, compute $\int_U fg$

so $-\Delta\varphi + \tilde{K} - 2\pi\chi(\Sigma)e^{2(\varphi+\gamma)} = 0$

i.e. φ solves $(**)$ and $f = \varphi + \gamma$ solves $(*)$

so $g' = e^{2f}g$ has Gauss curvature $2\pi\chi(\Sigma) < 0$

<u>note</u>: if g has constant Gauss curvature K

then cg, for $c>0$ a constant, has Gauss curvature $\frac{1}{c}K$

$\therefore$ any surface with $\chi(\Sigma)<0$, has a metric curvature being any negative constant.

<u>Case 3</u>: $\chi(\Sigma) > 0$

similar but harder

$\boxplus$

<u>Other Theorem (Kazdan-Warner)</u>:

Σ a compact surface with $\chi(\Sigma) < 0$ with a Riemonnian metric g

given a smooth function $C : \Sigma \to \mathbb{R}$ with $f(\vec{p}) \leq 0$ for all $\vec{p} \in \Sigma$ and < 0 at some $\vec{p}$

then there is some metric $g' = e^{2f}g$ such that

$$K'(\vec{p}) = f(\vec{p})$$