VII Mean Curvature

A Minimal Surfaces

a surface $\Sigma \subset \mathbb{R}^3$ is called minimal if

$$H(\vec{p}) = 0 \qquad \forall \vec{p} \in \Sigma$$

mean curvature

Why the name?

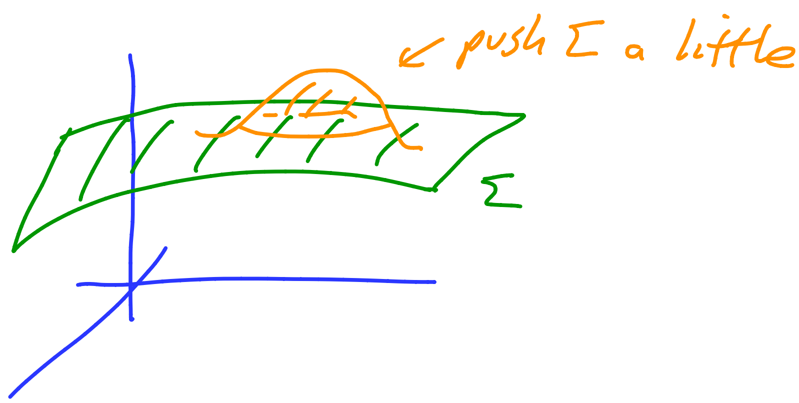
Recall: if R is a region in Σ then in Section IV B

we saw

$$A(R) = \int_R \sqrt{\det g}\; du\, dv$$

now given Σ in $\mathbb{R}^3$ let's consider surfaces

near Σ



Suppose $\vec{f} : V \to \Sigma$ is a coordinate chart

a normal variation of Σ is

$$\vec{\Phi} : V \times (-\varepsilon, \varepsilon) \to \Sigma$$

$$\vec{\phi}((u,v), t) = \vec{f}(u,v) + t\, h(u,v)\, \vec{N}(\vec{f}(u,v))$$

where $h : V \to \mathbb{R}$ some function and $h = 0$

outside a compact set $D \subset V$

and $\vec{N}$ is the normal vector to Σ

let $\vec{f}^{\,t} : V \to \mathbb{R}^3$ be $\vec{f}^{\,t}(u,v) = \vec{\Phi}((u,v), t)$

this gives near by parameterized surfaces

$$\vec{f}^{\,t}_u = \vec{f}_u + t\, h_u \vec{N} + t\, h \vec{N}_u$$

$$\vec{f}^{\,t}_v = \vec{f}_v + t\, h_v \vec{N} + t h \vec{N}_v$$

recall the second fundamental form is $\mathrm{II} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$$A = -\vec{f}_u \cdot \vec{N}_u \quad B = -\vec{f}_u \cdot \vec{N}_v \quad C = -\vec{f}_v \cdot \vec{N}_v$$

so $g^t_{11} = \vec{f}^{\,t}_u \cdot \vec{f}^{\,t}_u = f_u \cdot f_u + 2t\left(h\, \vec{f}_u \cdot \vec{N}_u + \cancelto{0}{h_u \vec{f}_u \cdot \vec{N}}\right)$

$$+ t^2 h^2 \vec{N}_u \cdot \vec{N}_u + 2t^2 h h_u \vec{N}_u \cdot \vec{N}$$

$$+ t^2 h_u^2\, \vec{N} \cdot \vec{N}$$

$$= g_{11} - 2t h A + t^2 (\text{—})$$

similarly

$$g_{22}^t = g_{22} - 2thC + t^2(\text{———})$$

$$g_{12}^t = g_{12} - 2thB + t^2(\text{———})$$

so $\det(g_{ij}^t) = g_{11}^t g_{22}^t - (g_{12}^t)^2$

$$= \begin{pmatrix} g_{11} - 2thA + t^2(\text{—}) & g_{12} - 2thB + t^2(\text{—}) \\ g_{12} - 2thB + t^2(\text{—}) & g_{22} - 2thC + t^2(\text{—}) \end{pmatrix}$$

$$= \det(g_{ij}) - 2th(Ag_{22} + Cg_{11} - 2g_{12}B) + t^2(\text{———})$$

recall $H = \frac{1}{2} \frac{g_{22}A + Cg_{11} - 2g_{12}B}{\det(g_{ij})}$

so $\det(g_{ij}^t) = \det(g_{ij})(1 - 4htH) + t^2(\text{—})$

and the area of $\vec{f}^t(D)$ is

$$A(t) = \int_D \sqrt{\det(g_{ij}^t)}\, du\, dv$$

$$= \int_D \sqrt{1 - 4htH + t^2(\text{—})}\, \sqrt{\det(g_{ij})}\, du\, dv$$

now

$$\frac{d}{dt} A(t)\Big|_{t=0} = \int_D \frac{1}{2}(\quad)^{1/2}\left(-4hH + 2t(-)\right)\sqrt{\det(g_{ij})}\,du\,dv\Big|_{t=0}$$

$$= \int_D -2hH\sqrt{\det(g_{ij})}\,du\,dv$$

so $A'(0) = -2\int_D hH\,dA$

Th$^{\underline{m}}$ 1:

$$\boxed{\Sigma \subset \mathbb{R}^3 \text{ is minimal} \Leftrightarrow A'(0)=0 \text{ for all such normal variations}}$$

Proof: ($\Rightarrow$) clear

($\Leftarrow$) if $H \neq 0$ near $\vec{p}$ then choose h such that $h(\vec{q}) = H(\vec{q})$ for $\vec{q}$ near $\vec{p}$ and zero elsewhere

then $A'(0) < 0$ $\emptyset$ ▦

Remark: so if Σ minimal then for any bounded region $\vec{f}(D)$ is a critical point for the area function

but not necessarily a minima!

Th^m 2:

$$\boxed{\text{If } \Sigma \subseteq \mathbb{R}^3 \text{ is compact without boundary then } \Sigma \text{ is not minimal}}$$

Proof: Theorem IV. 8 says $\exists \vec{p} \in \Sigma$ s.t. $K(\vec{p}) > 0$

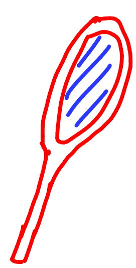
so K_1, K_2 have same sign and hence

$$H(\vec{p}) = \frac{1}{2}(K_1 + K_2) \neq 0$$

Physically what are minimal surfaces?

If you look at a soap film, the tension in the film will try to minimize the area of the film subject to whatever constraints might be placed of the film

e.g. consider the soap film on a "bubble wand"



it will have mean curvature 0 and minimize the area of a surface with given boundary

B Constant Mean Curvature

Consider a soap bubble: the tension in the soap film will try to minimize the area of the bubble making it smaller, but the air preasure inside is resisting being contracted. At some point the "preasure" equals the "tension" and you have a stable bubble

the tension is given by the mean curvature (recall where even the mean curvature is largest the area will change the most if one moves in the normal direction)

note sense the preasure and tension is uniform the mean curvature should be constant

Σ is called a surface of constant mean curvature (CMC) if $H(\vec{p})$ is constant

$$\boxed{\text{A soap bubble always takes on the form of a surface of constant mean curvature}}$$

Alexandrov's Th$^{\underline{m}}$ 3:

If Σ is a compact embedded surface in $\mathbb{R}^3$ of constant mean curvature, then Σ is a standard round sphere

Soap bubbles are spheres!

We need two lemmas

lemma 4:

If Σ is a compact embedded surface in $\mathbb{R}^3$ bounds a domain D of volume $V(D)$ and $H > 0$ then

$$\int_\Sigma \frac{1}{H}\, dA \geq 3V(D)$$

with equality $\Leftrightarrow$ Σ is standard round sphere

lemma 5:

If $\vec{v}$ is any vector field in $\mathbb{R}^3$ and we let

$$\phi((u,v),t) = \vec{f}(u,v) + t\,\vec{v}(\vec{f}(u,v))$$

be a "variation of a surface parameterized by $\vec{f}$

then $A'(0) = -\int_\Sigma 2H(\vec{v}\cdot\vec{N})\, dA$

Proof of 5: just like the computation of $A'(0)$ above but messier $\boxplus$

Proof of Th$^{\underline{m}}$ 3 assuming lemma 4:

let $\vec{v}(x,y,z) = (x,y,z)$ in the formula above

$$\phi((u,v),t) = \vec{f}(u,v) + t\vec{f}(u,v)$$
$$= \vec{f}(u,v)(1+t)$$

<u>note</u>: $\vec{f}_u^{\,t} = (1+t)\vec{f}_u$, $\vec{f}_v^{\,t} = (1+t)\vec{f}_v$

$$\therefore\ g_{ij}^t = (1+t)^2 g_{ij} \quad \text{and} \quad dA^t = (1-t)^2 dA$$

$$\text{so } A(t) = (1+t)^2 A \quad \text{and} \quad A'(t) = 2(1+t)A$$

$$\therefore\ A'(0) = 2A$$

$\leftarrow$ area(Σ)

from lemma 5

$$A'(0) = -\int 2H\, \vec{v}\cdot\vec{N}\, dA$$

choose $\vec{N}$ to be inward pointing

If H is constant

$$A = H\int \vec{v}\cdot\vec{N}\, dA$$
$$= H\int_D \operatorname{div}\vec{v}\ dx\,dy\,dz \qquad \partial D = \Sigma$$

divergence theorem from vector calc.

$$= H\int_D 3\, dx\,dy\,dz = 3H\,V(D)$$

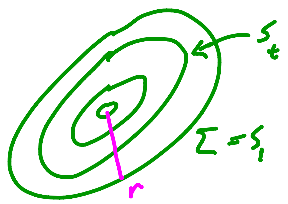
$$\text{now } \int \frac{1}{H} dA = \frac{1}{H}\int dA = \frac{A}{H} = \frac{A}{(A/3V(D))} = 3V(D)$$

$\therefore$ lemma 4 says Σ round sphere

Proof of lemma 4:

We begin by finding a clever way to compute the volume of D

the idea is:

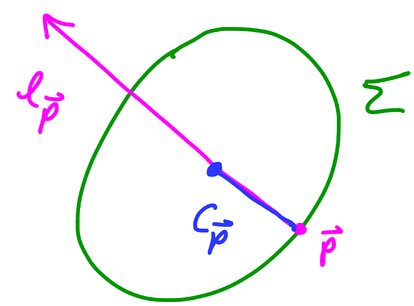


$$V = \int_0^r \text{area } S_t \, dt$$

"co-area formula"

now we are given Σ and N the inward pointing normal vector

for each $\vec{p} \in \Sigma$ consider the ray $l_{\vec{p}}$ through $\vec{p}$ in the direction of $\vec{N}(\vec{p})$



let $C_{\vec{p}} = \{ \vec{q} \in l_{\vec{p}}$ s.t. $\vec{q}$ closer to $\vec{p}$ than any other point on $\Sigma \}$

and set $d(\vec{p}) = \sup \{ \| \vec{p} - \vec{q} \| \text{ with } \vec{q} \in C_{\vec{p}} \}$

so for all $\vec{q} \in l_{\vec{p}}$ with $\| \vec{p} - \vec{q} \| < d(\vec{p})$, $\vec{p}$ is closest point on Σ to $\vec{q}$

now consider
$$\vec{f} : \Sigma \times [0,1] \to \mathbb{R}^3$$
$$(\vec{p}, t) \mapsto \vec{p} + t \left(d(\vec{p}) \vec{N}(\vec{p}) \right)$$

and let $S = \text{image}(\vec{f})$

Claim 1:

$$\boxed{S = D}$$ ← domain bounded by Σ

Claim 2:

$f|_{\Sigma \times [0,1)} : \Sigma \times [0,1) \to \mathbb{R}^3$ is

(1) injective

(2) $\det(Df) \neq 0$

Claim 3:

$$V(D) = Vol(\text{Image } \vec{f}) = Vol(\text{Image } \vec{f}|_{\Sigma \times [0,1)})$$

$$= \int_\Sigma \int_0^{d(\vec{p})} |1 - 2t H(\vec{p}) + t^2 K(\vec{p})| \, dt \, dA$$

mean curvature (H), Gauss curvature (K)

now if κ_1 and κ_2 are the principal curvatures at $\vec{p}$

then $1 - t2H + t^2 K = 1 - t(\kappa_1 + \kappa_2) + t^2 \kappa_1 \kappa_2$

$$= (1 - \kappa_1 t)(1 - \kappa_2 t)$$

so

$$V(D) = \int_\Sigma \int_0^{d(\vec{p})} |(1 - \kappa_1 t)(1 - \kappa_2 t)| \, dt \, dA$$

Claim 4:

$$\boxed{\frac{1}{d(\vec{p})} \geq \max\{K_1(\vec{p}), K_2(\vec{p})\}}$$

so $\frac{1}{d(\vec{p})} \geq K_1(\vec{p}) \Rightarrow 1 \geq d(\vec{p}) K_1(\vec{p})$

and similarly $1 \geq d(\vec{p}) K_2(\vec{p})$

$\therefore \quad K_1 t, \; K_2 t \leq 1 \quad \forall \; t \in [0, h(\vec{p})]$

and so

$$\int_0^{h(\vec{p})} |(1-K_1 t)(1-K_2 t)| dt = \int_0^{h(\vec{p})} (1-K_1 t)(1-K_2 t) dt$$

note: useful inequality $ab \leq \left(\frac{a+b}{2}\right)^2 \quad a, b \geq 0$

indeed: $(\sqrt{a} - \sqrt{b})^2 \geq 0$

$$\underset{a - 2\sqrt{ab} + b}{\overset{\shortparallel}{}}$$

so $2\sqrt{ab} \leq a+b$ and $ab \leq \left(\frac{a+b}{2}\right)^2$

get equality $\Leftrightarrow a=b$ ✓

$$\therefore \left(1 - \frac{1}{d(\vec{p})} t\right)^2 = \left(1 - \frac{1}{d(\vec{p})} t\right)\left(1 - \frac{1}{d(\vec{p})} t\right)$$

$$\leq (1-K_1 t)(1-K_2 t) = \left(\frac{1-K_1 t + 1 - K_2 t}{2}\right)^2$$

$$= (1 - H(\vec{p}) t)^2$$

also $H(\vec{p}) = \frac{1}{2}(\kappa_1 + \kappa_2) \leq \frac{1}{2} 2 \frac{1}{d(p)} = \frac{1}{d(p)}$

so $d(\vec{p}) \leq \frac{1}{H(\vec{p})}$

now $$\int_0^{d(\vec{p})} (1-\kappa_1 t)(1-\kappa_2 t)\,dt \leq \int_0^{d(\vec{p})} (1 - H(\vec{p})t)^2\,dt$$
$$\leq \int_0^{1/H(\vec{p})} (1 - H(\vec{p})t)^2\,dt$$
$$= -\frac{1}{3}(1 - H(\vec{p})t)^3 \frac{1}{H(\vec{p})} \Big|_0^{1/H(\vec{p})}$$
$$= 0 + \frac{1}{3H(\vec{p})} = \frac{1}{3H(\vec{p})}$$

$$\therefore V = \int_\Sigma \int_0^{d(\vec{p})} (1-\kappa_1 t)(1-\kappa_2 t)\,dt\,dA$$
$$\leq \int_\Sigma \frac{1}{3H(\vec{p})}\,dA$$

so $\int_\Sigma \frac{1}{H(\vec{p})}\,dA \geq 3V$ as claimed

if we have $=$ then need $1-\kappa_1 t = 1-\kappa_2 t$

$\therefore \kappa_1 = \kappa_2$ for all $\vec{p} \in \Sigma$

so Σ is totally umbilic

$\therefore \Sigma$ a round sphere by Th$^{\underline{m}}$ IV.7

Proof of Claim 1:

It is easy to see $S \subset D$

for the other inclusion let $\vec{q} \in D$

let $\vec{p} \in \Sigma$ be a point in Σ as close to $\vec{q}$ as any point in Σ

<u>note</u>: $\vec{q}$ lies on $\ell_{\vec{p}}$

since if not the line from $\vec{p}$ to $\vec{q}$ is not perpendicular to $T_{\vec{p}}\Sigma$

let $\vec{u} \in T_{\vec{p}}\Sigma$ with a component pointing toward $\vec{q}$ $\quad (\|\vec{u}\| = 1)$

let $\vec{\alpha} : (-\varepsilon, \varepsilon) \to \Sigma$ be unit speed curve with $\vec{\alpha}(0) = \vec{p}$ and $\vec{\alpha}'(0) = \vec{u}$

Consider $f(t) = |\vec{q} - \vec{\alpha}(t)|^2$

$$f'(t) = -2\,\vec{\alpha}'(t) \cdot (\vec{q} - \vec{\alpha}(t))$$
$$= -2\,\vec{\alpha}'(t) \cdot \vec{q}$$

(note: $\vec{\alpha}'(t) \cdot \vec{\alpha}(t) = 0$ since $\|\alpha'(t)\| = 1$)

so $f'(0) = -2\,\vec{u} \cdot \vec{q} < 0$

so $\vec{\alpha}(t)$ closer to $\vec{q}$ than $\vec{\alpha}(0) = \vec{p}$ for small positive t. ✓

now if $\vec{q} \in C_{\vec{p}}$ done since $\vec{q} = \vec{F}(\vec{p}, t)$ where
$$t = \frac{\|\vec{p} - \vec{q}\|}{d(\vec{p})}$$

if $\vec{q} \in C_{\vec{p}}$ then $\vec{q} \in \overline{C_p}$ i.e. a limit point of $C_{\vec{p}}$

so $\vec{q} = \vec{f}(\vec{p}, 1)$

Proof of Claim 2:

<u>injectivity:</u>

suppose $\vec{f}(\vec{p}, t) = \vec{q} = \vec{f}(\vec{p}', t')$ $\quad t, t' \in [0,1)$

then $\vec{q} \in C_{\vec{p}}$ and $\vec{q} \in C_{\vec{p}'}$

but $\vec{q} \in C_{\vec{p}}$ means $\vec{q}$ closer to $\vec{p}$ than

<u>any</u> other point in Σ $\therefore$ $\vec{p} \notin C_{\vec{p}'}$

unless $\vec{p} = \vec{p}'$

of course $\vec{f}(\vec{p}, t) = \vec{f}(\vec{p}, t') \Leftrightarrow \underline{t = t'}$ ✓

<u>for Def $D\vec{f}$:</u>

let $\vec{x} : V \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}^3$ be a coordinate chart around $\vec{p}$

then $\vec{y} : V \times (t - \varepsilon, t + \varepsilon) \to \mathbb{R}^3$

$$(u, v, t) \mapsto \vec{x}(u, v) + t \vec{N}(u, v)$$

is a representation of $\vec{f}$ in coordinate chart

here $\varepsilon > 0$ small enough s.t. all $s \in (t - \varepsilon, t + \varepsilon)$

are in $[0, d(\vec{p}))$

now compute:

$$D\vec{f} = D\vec{y} = \begin{bmatrix} \vec{x}_u + t\vec{N}_u & \vec{x}_v + t\vec{N}_v & \vec{N} \end{bmatrix}$$

recall for a 3×3 matrix $M = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3]$

$$\det M = (\vec{v}_1 \times \vec{v}_2)\cdot \vec{v}_3$$

recall, using the basis $\vec{x}_u$ and $\vec{x}_v$ we can write the shape operator $S_{\vec{p}}(\vec{w}) = -\vec{N}_{\vec{w}}$ as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{where } -\vec{N}_{\vec{x}_u} = a_{11}\vec{x}_u + a_{21}\vec{x}_v$$

$$-\vec{N}_{\vec{x}_v} = a_{12}\vec{x}_u + a_{22}\vec{x}_v$$

so

$$\det D\vec{f} = \left(\vec{x}_u \times \vec{x}_v + t(\vec{x}_u \times \vec{N}_v + \vec{N}_u \times \vec{x}_v) + t^2\, \vec{N}_u \times \vec{N}_v\right)\cdot \vec{N}$$

$$= \Big(\vec{x}_u \times \vec{x}_v + t\Big[\vec{x}_u \times \big(\cancelto{0}{-a_{12}\vec{x}_u} - a_{22}\vec{x}_v\big) +$$

$$\big(-a_{11}\vec{x}_u \cancelto{0}{- a_{21}\vec{x}_v}\big)\times \vec{x}_v\Big]$$

$$+ t^2\Big[(-a_{11}\vec{x}_u - a_{21}\vec{x}_v)\times(-a_{12}\vec{x}_u - a_{22}\vec{x}_v)\Big]\Big)\cdot\vec{N}$$

$$= \Big(\vec{x}_u \times \vec{x}_v - t\,(a_{11} + a_{22})\,\vec{x}_u \times \vec{x}_v$$

$$+ t^2(a_{11}a_{22} - a_{12}a_{21})\,\vec{x}_u \times \vec{x}_v\Big)\cdot \vec{N}$$

but recall $\vec{x}_u, \vec{x}_v$ span $T_{\vec{q}}\Sigma$ so

$\vec{x}_u \times \vec{x}_v$ normal to $T_{\vec{q}}\Sigma$

so if $\vec{v} \perp T_q \Sigma$ then $\vec{v} \cdot \vec{N} = \pm \|\vec{v}\|$

$$= \pm \|\vec{x}_u \times \vec{x}_v\| \, |1 - t\,2H + t^2 K|$$

as observed above if K_1, K_2 principal curvatures we have

$$= \pm \|\vec{x}_u \times \vec{x}_v\| \, |(1 - tK_1)(1 - tK_2)|$$

after Claim 4 we observed

$$K_1 t, K_2 t < 1 \quad \forall\, t \in [0, d(\vec{p}))$$

so $\det D\vec{f} > 0$ $\blacksquare$

Proof of Claim 3:

we can compute the volume of image $\vec{y}$ by

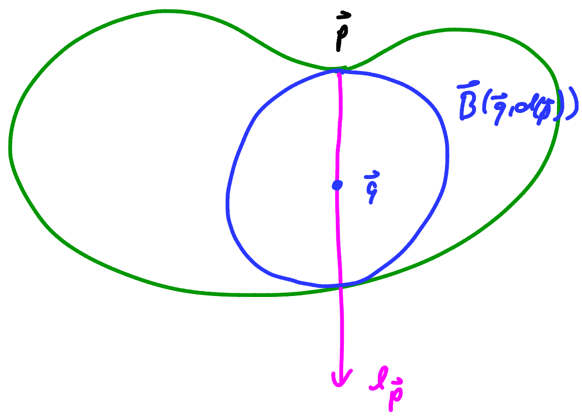
$$\int |\vec{x}_u \times \vec{x}_v| \, |1 - t\,2H + t^2 K| \, dt\, du\, dv$$

$$= \int_V \int_{t-\varepsilon}^{t+\varepsilon} |1 - t2H + t^2 K| \, dt\, dA$$

since we can do this for every coord chart we get the formula in the claim $\blacksquare$

Proof of Claim 4:

let $\vec{q}$ be a point on $l_{\vec{p}}$ a distance $d(\vec{p})$ from $\vec{p}$ and $B(\vec{q}, d(\vec{p}))$ be a ball of radius $d(\vec{p})$ about $\vec{q}$



then the interior of $B(\vec{q}, d(\vec{p}))$ contains no points of Σ on its interior

<u>note</u>: $\partial B(\vec{q}, d(\vec{p}))$ is a sphere of radius $d(\vec{p})$ so all of its normal curvatures are $\frac{1}{d(\vec{p})}$

now arguing exactly as in the proof of Thm IV.8 we see that the principal curvatures of Σ at $\vec{p}$ are $\leq$ those of $\partial B(\vec{q}, d(\vec{p}))$ ▦