Math 501 - Spring 2002
Final Exam

Work 7 of the 10 problems.

1. In $\mathbb{R}^3$ let (ρ, θ, ϕ) be spherical coordinates. Let $\gamma(s) = (1, \theta(s), \phi(s))$ be a curve on the unit sphere parameterized by arc length and assume that $\gamma'(0) = (0, 1, 0)$. Show that

$$\kappa(0) = \sqrt{1 + (\phi''(0))^2}.$$

2. Let C be a regular plane curve which lies outside a disk of radius R. Show that there is a point on C where the curvature is $\leq \frac{1}{R}$.

3. Suppose a plane curve C is given in polar coordinates by $r = f(\theta)$. Show that the arc length along C is given by

$$\int_0^b \sqrt{r^2 + (r')^2} d\theta,$$

and the curvature is

$$\kappa(\theta) = \frac{2(r')^2 - rr'' + r^2}{((r')^2 + r^2)^{\frac{3}{2}}},$$

where $r' = \frac{dr}{d\theta}$.

4. Suppose two Riemannian metrics g and g' on a surface Σ are related by $g' = e^f g$ for some function $f : \Sigma \to \mathbb{R}$. Compute the Gauss curvature K' of g' in terms of the Gauss curvature K of g and f.

5. Let $\Sigma = \{z = x^3 - 3y^2 x\}$. Compute the shape operator and the first fundamental form of Σ at the origin. Compute the Gauss and mean curvature.

6. Suppose that the first fundamental form of Σ in local coordinates (u, v) is

$$\begin{pmatrix} 1 & 2\cos(f) \\ 2\cos(f) & 1 \end{pmatrix},$$

where f is a function of u and v. Show

$$K = \frac{-f_{uv}}{\sin f}.$$

7. Given a (regular) curve C in the xy-plane we can parametrize it (by arc length) by $\alpha(s) = (g(s), h(s), 0)$. If we rotate C about the x-axis then the we will get a surface Σ, called the surface of revolution. A parameterization of Σ is given by $\mathbf{x}(s, \theta) = (g(s), h(s)\cos\theta, h(s)\sin\theta)$. Show the first fundamental form is

$$\begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix},$$

where $R = R(s)$ is the distance of $\alpha(s)$ form the x-axis. Find the Christoffel symbols. Write down the equations that a geodesic must satisfy. Are there any obvious solutions?

8. Let $H^2 = \{(x, y) \colon y > 0\}$ with the Riemannian metric

$$\frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let X and Y be the vector fields $X = e^y \frac{\partial}{\partial x}$ and $Y = e^x \frac{\partial}{\partial y}$, respectively. Calculate the covariant derivatives$\nabla_X Y$, $\nabla_Y X$.

9. Let Σ be a Riemannian surface with metric $\langle , \rangle$ and let $f \colon \Sigma \to \mathbb{R}$ be a smooth function. The vector field ∇f is defined as the unique solution to the equations

$$\langle \nabla f(p), V \rangle = df(V) \quad \text{for all } V \in T_p\Sigma.$$

An integral curve of a vector field X on Σ is a curve $c(t)$ such that $\frac{dc(t)}{dt} = X(c(t))$. Show that if $\langle \nabla f, \nabla f \rangle \equiv 1$ then the integral curves of ∇f are geodesics.

10. Σ is called *symmetric* if for every point $p \in \Sigma$ there exists a neighborhood U_p of p in Σ and an isometry $f_p \colon \Sigma \to \Sigma$ such that $f_p(p) = p$, $f_p(u) \neq u$ for all $u \in U_p - \{p\}$ and such that $f_p^2(x) = f_p(f_p(x)) = x$ for all $x \in \Sigma$.

   (a) Show that S^2, the unit sphere in $\mathbb{R}^3$ with the induced metric, is symmetric.

   (b) Let Σ be symmetric. Let p be a point in Σ and f_p its associated isometry. Show that if $v \in T_p\Sigma$ then $df_p(v) = -v$.

   (c) Let $c \colon [0, l] \to \Sigma$ be a geodesic in a symmetric Riemannian surface. Show that there is an isometry $g \colon \Sigma \to \Sigma$ such that $g(c(0)) = g(c(l))$.