Math 501 - Spring 2002
Homework 1

PART I

Work 5 of the 8 problems in this section.

1. Give a formula for the arc length along an ellipse. Look up the integral in a standard table. What is the integral called? Any guesses as to why? What does this mean concerning parameterizing the ellipse by arc length?
2. Let $\alpha(t) = (a\cos^2 t, a\sin t\cos t, a\sin t)$. Find the curvature of α.
3. Compute the curvature and torsion of the curve $\beta(s) = (\frac{1}{\sqrt{2}}\cos s, \sin s, \frac{1}{\sqrt{2}}\cos s)$. Identify the curve.
4. Find the curvature of the ellipse: $\alpha(t) = (a\cos t, b\sin t)$.
5. Give a formula, in terms of $x(t)$ and $y(t)$ and their derivatives, for the curvature for a curve in $\mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t))$.
6. Give a formula, in terms of f and its derivative, for the curvature of a curve in $\mathbb{R}^3$ given by $\{(x, y, z) : y = x, z = f(x)\}$.
7. Let $\beta(s) = (\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}})$ for $s \in [-1, 1]$. Show that β is unit speed. Find the curvature and torsion of β.
8. Find the parameterization by arc length $\beta(s)$ of a plane curve with $\kappa(s) = \frac{1}{1+s^2}$ and $\beta(0) = (0, 0)$.

PART II

Work 4 of the 7 problems in this section.

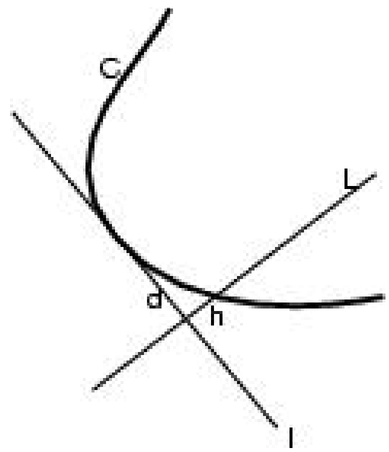
9. Let l_p be the tangent line to a curve C and p. Let L be a line perpendicular to l_p and a distance d from p. Let h be the length of the line segment on L between l_p and C (h is the height of C relative to l_p). See Figure Below. Prove

$$\kappa(p) = \lim_{d\to 0} \frac{2h}{d^2}.$$

10. Let C be a plane curve parameterized by arc length by $\alpha(s)$, $T(s)$ its unit tangent vector and $N(s)$ be its unit normal vector. Show

$$\frac{d}{ds}N(s) = -\kappa(s)T(s).$$

11. Let C be a plane curve contained in a disk of radius R. Show there is a point on C at which the curvature is $\geq \frac{1}{R}$.
HINT: You can assume the disk is centered at the origin, then think about the largest r for which the circle of radius r intersects C. You might also find Problem 8. useful here.



12. Let $\alpha(t)$ parameterize a simple closed curve C. Assume that $0 < K(t) \leq c$ where c is some constant. (Recall K is the signed curvature of a plane curve.) Prove

$$\text{length } C \geq \frac{2\pi}{c}.$$

13. Show that if α is a parameterization by arc length of a curve lying on a sphere of radius R about the origin in $\mathbb{R}^3$ then

$$R^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2.$$

HINT: Write $\alpha(s) = a(s)T(s) + b(s)N(s) + c(s)B(s)$ (where T, N, B are the unit tangent, normal and binormal vectors) and try to determine what a, b, c are. To do this consider $\alpha \cdot \alpha = R^2$. Differentiate this several time and see what relations you get.

14. Show that if α is a parameterization by arc length of a curve and $\kappa'(s) \neq 0$ $\tau(s) \neq 0$ and they satisfy the previous equation for some R then the image of α lies on some sphere of radius R. HINT: Think about your solution to the last problem.

15. Let $\alpha(s)$ and $\beta(s)$ be two unit speed curves. Assume
$\alpha(0) = \beta(0)$,
$\alpha'(0) = \beta'(0)$,
$\kappa_\alpha(s) = \kappa_\beta(s)$ and
$\tau_\alpha(s) = \tau_\beta(s)$.
Let

$$D(s) = T_\alpha(s) \cdot T_\beta(s) + N_\alpha(s) \cdot N_\beta(s) + B_\alpha(s) \cdot B_\beta(s).$$

Show
a) $D(0) = 3$,
b) $D(s) = 3$ implies the Frenet frames of α and β agree at s,
c) $D'(s) = 0$ (and thus $D(s) = 3$),
d) $\alpha(s) = \beta(s)$ for all s.
Note you have just shown that a (biregular) curve in $\mathbb{R}^3$ is uniquely determined (up to translation and rotation) by its curvature and torsion.