Math 501 - Spring 2002
Homework 2

PART I

Work 4 of the 7 problems in this section.

1. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:
   a) The paraboloid: $\{(x, y, z) \in (R)^3 : z = x^2 + y^2\}$,
   b) The torus: the image of the map $f(u, v) = (a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u)$,
   c) The Cateniod: $\{(x, y, z) \in (R)^3 : x^2 + y^2 = \cosh^2 z\}$.
2. Compute the Gauss and mean curvatures of the hyperboloid $z = axy$ at $(0, 0, 0)$.
3. Compute the principal, Gauss and mean curvatures of the Enneper surface: that is the image of the map $\mathbf{x}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$.
4. Let $\Gamma_f = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\}$ be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Show

$$H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{(1 + f_x^2 + f_y^2)^2},$$

   and

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

5. Given a (regular) curve C in the xy-plane we can parametrize it by $\alpha(u) = (g(u), h(u), 0)$. If we rotate C about the x-axis then the we will get a surface Σ, called the surface of revolution. A parameterization of Σ is given by $\mathbf{x}(u, v) = (g(u), h(u)\cos v, h(u)\sin v)$. Find formulas for the Gauss and mean curvatures in terms of g, h and their derivatives.
6. Let S^2 be the unit two sphere in $\mathbb{R}^3$. We define a map from $S^2 \setminus \{N\}$ to $\mathbb{R}^2$, where $N = (0, 0, 1)$ is the north pole. Given any point $p \in S^2$ let L_p be the line containing the points p and N. The line L_p will intersect the xy-plane in a unique point q. We define $f(p) = q$ and call the map $f : S^2 \setminus \{N\} \to \mathbb{R}^2$ stereographic projection.
   a) If we write $p = (x, y, z)$ find a formula for L_p.
   b) Find a formula for $f(p) = f(x, y, z)$, in terms of x, y, z.
   c) Let $g(u, v) = f^{-1}(u, v)$. Find a formula for $g(u, v)$.
   d) Note g is a coordinate chart for S^2. Compute the Gauss curvature using this coordinate chart.
   e) Compute the components of the first fundamental form in this coordinate chart.

   Note if we also defined stereographic projection from the south pole we would be able to cover S^2 but two coordinate charts instead of the six we needed in class.
7. Let $\alpha(t) = (\cos t + \log(\tan\frac{t}{2}), \sin t)$ be a curve in $\mathbb{R}^2$, here $t \in (0, \pi)$. Let S be the surface of revolution of α. Show that $K = -1$.

PART II

Work 3 of the 4 problems in this section.

8. Let $h : \Sigma \to \mathbb{R}$ be a smooth function on Σ and $p \in \Sigma$. Recall the derivative of h is $Dh_p : T_p\Sigma \to \mathbb{R}$ is just $D_v h$ for $v \in T_p\Sigma$ (i.e. Dh is just the directional derivative of h in directions tangent to Σ). The point p is a critical point if $Dh_p = 0$. Assume p is a critical point, now take a vector $v \in T_p\Sigma$ and let $\alpha : (-\epsilon, \epsilon) \to \Sigma$ be a parameterized curve with $\alpha(0) = p$ and $\alpha'(0) = v$. Now set

$$H_p h(v) = \frac{d^2(h \circ \alpha)}{dt^2}|_{t=0}.$$

a) Let $\mathbf{x} : V \to \Sigma$ be a coordinate chart for Σ at p, and show that

$$H_p h(a\mathbf{x}_u + b\mathbf{x}_v) = h_{uu}a^2 + 2h_{uv}ab + h_{vv}b^2.$$

Note this implies that $H_p h$ is a well defined map from $T_p\Sigma$ to $\mathbb{R}$. This map is called the Hessian of h at p.
b) Let $h : \Sigma \to \mathbb{R}$ be the height function of Σ relative to $T_p\Sigma$, that is $h(q) = (q-p)\cdot N(p)$ for $q \in \Sigma$. Show p is a critical point of h. Then show

$$H_p h(v) = \kappa_p(v).$$

9. Suppose that $\mathbf{x}(u, v)$ is an isothermal coordinate chart for Σ. Recall this means that $E = G = \lambda(u, v)$ and $F = 0$. Show that

$$K = \frac{1}{2\lambda}\Delta(\log \lambda),$$

where $\Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$ is the Laplacian of f.

10. Suppose that $\mathbf{x}(u, v)$ is an isothermal coordinate chart for Σ. Show that

$$H = \frac{1}{2\lambda}\Delta\mathbf{x}.$$

11. Show that the mean curvature of Σ at a point p is given by

$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\mathbf{v}(\theta))d\theta,$$

where $\mathbf{v}(\theta)$ is the unit tangent vector in $T_p\Sigma$ obtained from a fixed vector $\mathbf{v}_0$ by rotation θ counterclockwise.