Math 501 - Spring 2002
Homework 3

PART I

Work 4 of the 7 problems in this section.

1. Compute the Gauss curvature of the following Riemannian metrics.

   a) (The spherical metric)
   $$g = \frac{4}{(1+u^2+v^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
   on $\mathbb{R}^2$.

   b) (The hyperbolic metric)
   $$g = \frac{4}{(1-(u^2+v^2))^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
   on the (open) unit disk $D^2 = \{(u,v) \in \mathbb{R}^2 | u^2+v^2 < 1\}$.

   c) (Hyperbolic metric again)
   $$g = \frac{1}{v^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
   on the upper half plane $H = \{(u,v) \in \mathbb{R}^2 | v > 0\}$.

2. Using the Riemannian metric in 1) b) compute:

   a) The area of $D_\rho = \{(u,v) | u^2+v^2 \leq \rho^2\}$.

   b) The length of the curve ∂D_ρ.

   c) Make the same computations using the metric from 1) a).

3. Consider the metric from 1) c). Prove that the vertical lines $u = c$ are geodesics. Prove that the intersection of any circle centered on the x-axis with H is a geodesic.

4. Recall a fractional linear transformation is a map of the form
   $$f(x) = \frac{az+b}{cz+d}$$
   where $z = u+iv$ and a, b, c, d are real numbers. Find conditions on a, b, c, d that ensure that f will map H to H (H is as in 1) c)). Show that all these fractional linear transformation are (local) isometries of the Riemannian metric in 1) c).

5. Compute the total curvature of the the unit sphere S^2 in $\mathbb{R}^3$, that is compute
   $$\int_{S^2} K(p)dA$$
   where $K(p)$ is the Gauss curvature.
   Hint: If you use stereographic coordinates you can cover S^2 minus a point with one coordinate chart, and therefor do all your computation in this coordinate chart. Recall the first fundamental form in stereographic coordinates is given in 1) a).

6. We have discussed two ways of putting a metric on T^2. One in $\mathbb{R}^3$ as the image of

$$\overline{x}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$

where $0 < r < a$ are two preassigned constants. And one in $\mathbb{R}^4$ as the image of

$$\overline{y}(u,v) = (\cos u, \sin u, \cos v, \sin v).$$

Denote the Riemannian metric coming from the first representation by g and from the second by h. Compute the total curvature for both these metrics.

7. Let $\alpha : (-\epsilon, \epsilon) \to \Sigma$ be a regular arc in Σ. For $t \in (-\epsilon, \epsilon)$ let $\overline{w}(t) \in T_{\alpha(t)}\Sigma$, so $\overline{w}$ is a vector field along α. Define

$$\frac{D\overline{w}}{dt}(t) = (\nabla_{\alpha'(t)}\overline{w})(t).$$

If $\overline{w}$ and $\overline{v}$ are two vector fields along α show

$$\frac{d}{dt}g(\overline{v}(t), \overline{w}(t)) = g(\frac{D\overline{v}}{dt}(t), \overline{w}(t)) + g(\overline{v}(t), \frac{D\overline{w}}{dt}(t)).$$

PART II

Work 3 of the 4 problems in this section.

A diffeomorphisms $f : \Sigma \to \Sigma'$ is a one-to-one, onto smooth map with a smooth inverse. An isometry is a local isometry that is also a diffeomorphism.

8. Show that a diffeomorphism $f : \Sigma \to \Sigma'$ is an isometry if and only if the arc length of any regular curve in Σ is equal to the arc length of the image of the curve under f.

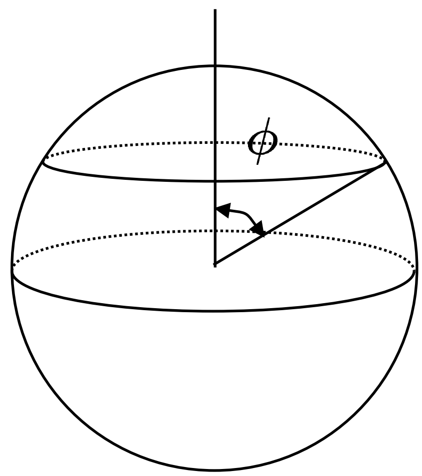
9. Let Σ, Σ' and Σ'' be three Riemannian surfaces. Prove
   a) If $f : \Sigma \to \Sigma'$ is an isometry then $f^{-1} : \Sigma' \to \Sigma$ is an isometry.
   b) If $f : \Sigma \to \Sigma'$ and $g : \Sigma' \to \Sigma''$ are isometries then so is $g \circ f$.

Note that this implies that the set of isometries of a fixed Riemannian surface is a group under composition, which we denote $Isom(\Sigma)$.

10. Let Σ be a surface in $\mathbb{R}^3$ and give Σ the Riemannian metric induced from $\mathbb{R}^3$.
    a) Show that if $F : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ and $F(\Sigma) = \Sigma$, then $F|_\Sigma : \Sigma \to \Sigma$ is an isometry of Σ.
    b) Show that the group of orthogonal linear transformations of $\mathbb{R}^3$ is a subgroup of the group of isometries of S^2 (the unit sphere). Recall $O(3)$, the group of orthogonal linear transformations of $\mathbb{R}^3$, is the set of 3×3 matrices with determinant ± 1.

11. Show $O(3) = Isom(S^2)$ where S^2 is given the round metric induced on it as the unit sphere in $\mathbb{R}^3$.
    Hint: Given any isometry f find an element A of $O(3)$ such that $A \circ f$ preserves a point p and $D(A \circ f)$ is the identity on T_pS^2. Then thinking about what isometries and geodesics are prove that $A = f^{-1}$.

12. Let N be the north pole of the unit sphere S^2 and p, q be two points on the equator such that the great circles through N and p and the great circle through N and q make



an angle of θ at N. Let v be a unit vector in $T_N S^2$. Parallel translate v along the great circle to p then along the equator to q then along the other great circle back to N. The end result of this is a new vector v'.

a) Determine the angle of v with v'.

b) Do the same thing when the points p, q instead of being on the equator are taken on a parallel of "colatitude ϕ." See the Figure.