Math 618 - Fall 2004
Homework 3

1. Given (X, A) a CW-pair of spaces with X path connected. Let x_0 be a base point in X and set $\Lambda(A)$ to be the space of paths in X beginning at x_0 and ending at a point in A. That is $\Lambda(A) = \{u \in C^0([0,1], X) : u(0) = x_0, u(1) \in A.\}$ Show
$$\pi_n(X, A, a_0) = \pi_{n-1}(\Lambda, \gamma_0)$$
where $a_0 \in A$ and γ_0 is any fixed arbitrary path from x_0 to a_0.

2. Suppose the fibration $p : E \to B$ has a section $s : B \to E$ (that is $p \circ s = id_B$). Show
$$\pi_n(E, e_0) = \pi_n(B, b_0) + \pi_n(F, e_0)$$
where $e_0 = s(b_0)$ and b_0 is a base point in B.

3. Compute the homotopy groups of $\mathbb{C}P^\infty$. The easiest way to do this is to think of S^∞ as an S^1 bundle over $\mathbb{C}P^\infty$. Note this shows that $\mathbb{C}P^\infty$ is a $K(\pi, n)$ for some π and n.

4. Show that any map $f : X \to Y$ from an n-dimensional CW complex to an n-connected space is null-homotopic.

5. A space X is *aspherical* if $\pi_n(X) = 0$ for $n > 1$. If Y is an aspherical space then show that for any homomorphism $\psi : \pi_1(X) \to \pi_1(Y)$ there is a map $f_\psi : X \to Y$ that induces ψ on π_1. In other words there is a one to one correspondence
$$[X, Y]_0 = Hom(\pi_1(X), \pi_1(Y)).$$

6. Prove the naturality of the primary obstruction. (That is prove the naturality part of Theorem III.4.)

7. Given a simply connected space X show there are spaces X_i and maps $f_i : X \to X_i$ with the following properties:
   (a) $\pi_j(X_i) = 0$ for $j > i$
   (b) $(f_i)_* : \pi_j(X) \to \pi_j(X_i)$ is an isomorphism for all $j \leq i$
   (c) there are fibrations $p_i : X_i \to X_{i-1}$ such that $p_i \circ f_i = f_{i-1}$ (What is the fiber of p_i?)