Math 618 - Fall 2004
Homework 4

1. Let M be a closed oriented n manifold. Show $H_{n-1}(M;\mathbb{Z})$ is torsion free.

2. If M is a closed orientable 3 manifold with $H_1(M;\mathbb{Z}) = 0$ show $H_*(M;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ (that is show M is a homology sphere). If you prefer a little challenge show that a closed oriented 3 manifold with $\pi_1(M) = 1$ is a homotopy sphere.

3. Let M be a compact orientable 3 manifold with boundary having $H_1(M)$ finite. Show ∂M is a union of spheres.

4. If M is a closed orientable $4k$ manifold then the cup product pairing on $H^{2k}(M;\mathbb{Z})$ is a symmetric nondegenerate paring. Picking a basis for $H^{2k}(M)$ the pairing can be represented by a matrix. Such a symmetric matrix can be diagonalized over $\mathbb{R}$. After diagonalizing the number of positive elements down the diagonal is called b_+^{2k} and the number of negative elements is b_-^{2k}. The signature of the pairing is $\sigma(M) = b_+^{2k} - b_-^{2k}$. Prove that if $M = \partial W$ for a compact oriented $4k+1$ manifold then $\sigma(M) = 0$.

5. Is $\mathbb{C}P^2 \# \mathbb{C}P^2$ the boundary of a compact oriented 5 manifold? For a challenge what about $\mathbb{C}P^2 \# -\mathbb{C}P^2$? Where $-\mathbb{C}P^2$ is $\mathbb{C}P^2$ with the reversed orientation.

6. Let W be a k-disk bundle over N, where N is a manifold. Show
$$H_p(N) = H_{k+p}(W, \partial W).$$

7. Compute the cohomology group of the loop spaces ΩS^2 and ΩS^3. (Hint: Use the path space fibration.)

8. Compute the cohomology ring of $\mathbb{R}P^\infty$. (Hint: Use the S^1 fibration of $\mathbb{R}P^\infty$ over $\mathbb{C}P^\infty$. You also might need to explicitly compute the first couple of cohomology groups by hand.)

9. Compute the first six cohomology groups of $K(\mathbb{Z}_2, 2)$.

10. Suppose $\{E_r, d_r\}$ is a first quadrant spectral sequence (*i.e.* $E_r^{s,t} = 0$ if s or t is less than zero), of cohomological type, converging to H^*. If $E_2^{s,t} = 0$ unless $s = 0$ or $s = n$ for some $n \geq 2$, then show
$$\cdots \to H^k \to E_2^{0,k} \to E_2^{n,k-n+1} \to H^{k+1} \to E_2^{0,k+1} \to \cdots.$$
What does this say about fibrations over S^n? This long exact sequence is called the Wang sequence.