# Math 6441 - Spring 2018
# Homework 1

Read Chapter 0 in Hatcher.
*Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in* **only** *problems 1, 2, 4, 5, 7.* **Due: In class on January 19.**

1. Given a topological space $X$ and points $x, y \in X$, let $e_x : \{*\} \to X$ be the function from the 1 point set $\{*\}$ that sends $*$ to $x$, and similarly for $e_y : \{*\} \to X$. Show that $x$ and $y$ are connected by a path if and only if $e_x$ is homotopic to $e_y$.
2. Show that a space $X$ is contractible if and only if every map $f : X \to Y$, for arbitrary $Y$, is null-homotopic if and only if every map $f : Y \to X$, for arbitrary $Y$, is null-homotopic.
3. Show that spaces $X$ and $Y$ are homotopy equivalent if and only if for any space $Z$ there is a one-to-one correspondence
$$\phi_Z : [X, Z] \to [Y, Z]$$
such that for all continuous maps $h : Z \to Z'$ we have $h_* \circ \phi_Z = \phi_{Z'} \circ h_*$ where $h_*$ is the map induced on homotopy classes of maps, that is the following diagram commutes
$$\begin{array}{ccc} [X, Z] & \xrightarrow{\phi_Z} & [Y, Z] \\ \big\downarrow{\scriptstyle h_*} & & \big\downarrow{\scriptstyle h_*} \\ [X, Z'] & \xrightarrow{\phi_{Z'}} & [Y, Z']. \end{array}$$
4. Show that $S^n * S^m = S^{n+m+1}$ where $X * Y$ stands for the join of $X$ and $Y$ (see Hatcher Chapter 0 for the definition). Hint: work this out for $n, m$ equal to 0 or 1 first.
5. In Chapter 0 of Hatcher's book he describes a CW structure on $S^\infty$ with two cells of each dimension, enumerate all the subcomplexes of $S^\infty$ with that CW structure.
6. Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $f \circ g \sim id_Y$ and $h \circ f \sim id_X$. More generally, show that $f$ is a homotopy equivalence if $f \circ g$ and $h \circ f$ are homotopy equivalences.
7. Given CW complexes X and Y, explicitly describe the CW structure on $X \times Y$ in terms of the CW structures on $X$ and $Y$. (That is describe the cells of $X \times Y$ and describe the attaching maps.) Choose a cell structure on $S^1$ and explicitly write out the CW structure on $S^1 \times S^1$ coming from the product structure.
8. If $G$ is a group and a topological space, then it is called a *topological group* if the maps
$$\mu : G \times G \to G : (g, h) \mapsto g \cdot h$$
and
$$i : G \to G : g \mapsto g^{-1}$$
are continuous, where $g \cdot h$ is the multiplication operation in $G$. Examples of topological groups include $\mathbb{R}$, $S^1$ (the unit circle in $\mathbb{C}$), and the group of $n \times n$ matrices with real (or complex) entries. If $G$ is a topological group then show that for any space $X$ the set $[X, G]$ is a group and for any continuous map $f : X \to Y$ the map $f^* : [Y, G] \to [X, G]$ defined in class is a homomorphism.
Fact: $[X, S^1]$ is the first cohomology group $H^1(X)$, though our definition later will be quite different.